Recent Applications of Proof Theory in Fixed Point and Ergodic Theory

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Corrected version Nov.20: a confused slide on the functional interpretation of weak compactness as well as a slide stating a bound on Browder’s theorem have been deleted as the latter has been superseded meanwhile: weak compactness can be bypassed resulting in a primitive recursive bound.
Logical analysis of proofs $P$ of conclusions

**Goal:** Additional information on $C$: **effective bounds**, **elimination of assumptions** (e.g. compactness).
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- **interpret** the formulas $A$ in $P$: $A \leftrightarrow A^\mathcal{I}$,
- interpretation $C^\mathcal{I}$ contains the **additional information**,
- construct by **recursion on $P$** a new proof $P^\mathcal{I}$ of $C^\mathcal{I}$. 
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Our approach is based on novel forms and extensions of:

**K. Gödel’s functional interpretation!**
Proof interpretations as tool for generalizing proofs

$P \xrightarrow{\mathcal{I}} P^\mathcal{I}$

$G \downarrow \quad \downarrow \mathcal{I}^G$

$P^G \xrightarrow{G^\mathcal{I}} (P^\mathcal{I})^G = (P^G)^\mathcal{I}$

- Generalization $(P^\mathcal{I})^G$ of $P^\mathcal{I}$: easy!
- Generalization $P^G$ of $P$: difficult!

T. Tao: $P =$ ‘soft analysis’, $P^\mathcal{I} =$ ‘hard analysis’. 
Consider

\[ A \equiv \forall x \exists y \forall z \ A_{qf}(x, y, z), \quad A_{qf} \text{ quantifier-free}. \]

**Example:**

\[ \forall k \exists n \forall m (|r_n - r_{n+m}| \leq 2^{-k}). \]

where \((r_n)\) is a nonincreasing sequence in \([0, 1] \cap \mathbb{Q}\).
Monotone convergence principle (PCM)

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1. Naive attempt try to find \(f\) with

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Problem: no computable \(f\) (E. Specker 1949).
2. Attempt: no-counterexample interpretation

Change

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to the equivalent **Herbrand normal form**

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We now ask for $$\Phi : \mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}$$ s.t.

$$\forall x, g A_{qf}(x, \Phi(x, g), g(\Phi(x, g)))$$

(no-counterexample interpretation (n.c.i.), G.Kreisel).
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**Solvable:** Let \( \tilde{g}(n) := n + g(n) \).

\[ \Phi((r_n), k, g) := \min y \leq \max_{i \leq 2^k - 1} (\tilde{g}^{(i)}(0)) \left( |r_y - r_{\tilde{g}(y)}| \leq 2^{-k} \right). \]
Problems of the no-counterexample interpretation

N.c.i. weak enough to ensure an effective solution but except for $\forall \exists \forall$-sentences $A$ too weak to provide the correct computational contribution of $A$ in given proofs.

Example: Infinitary Pigeonhole Principle (IPP):

$\forall n \in \mathbb{N} \ \forall f : \mathbb{N} \rightarrow \mathbb{C}^n \ \exists i \leq n \ \forall k \in \mathbb{N} \ \exists m \geq k (f(m) = i)$,

where $\mathbb{C}^n := \{0, 1, \ldots, n\}$. IPP is strictly in between $\exists$- and $\exists \forall$-induction.

In particular: use of IPP can cause arbitrary primitive recursive complexity, but it has a trivial n.c.i.
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**Example: Infinitary Pigeonhole Principle (IPP):**

$$\forall n \in \mathbb{N} \forall f : \mathbb{N} \to C_n \exists i \leq n \forall k \in \mathbb{N} \exists m \geq k \left( f(m) = i \right),$$

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In particular: use of IPP can cause arbitrary primitive recursive complexity, but it has a trivial n.c.i.
Gödel’s functional interpretation \( D \) combined with Krivine’s negative translation \( N \) results in an interpretation \( Sh = D \circ N \) (Streicher/K.07)

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A \mapsto A^{Sh} \quad \text{(Shoenfield variant)}
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such that

- $A^{Sh} \equiv \forall x \exists y \ A_{Sh}(x, y)$, where $A_{qf}$ is quantifier-free,
Gödel’s functional interpretation in five minutes

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- $A \leftrightarrow A^{Sh}$ by classical logic and quantifier-free choice in all types

$$\text{QF-AC} : \forall a \exists b \, F_{qf}(a, b) \rightarrow \exists B \forall a \, F_{qf}(a, B(a)).$$
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$$\text{QF-AC} : \quad \forall a \exists b \quad F_{qf}(a, b) \rightarrow \exists B \forall a \quad F_{qf}(a, B(a)).$$

- $x, y$ are tuples of functionals of finite type over the base types of the system at hand,
\[ A^{Sh} \equiv \forall u \exists x A_{sh}(u, x) \], \quad B^{Sh} \equiv \forall v \exists y B_{sh}(v, y). \]

**Sh1**  \( P^{Sh} \equiv P \equiv P_{sh} \) for atomic \( P \)

**Sh2**  \( (\neg A)^{Sh} \equiv \forall f \exists u \neg A_{sh}(u, f(u)) \)

**Sh3**  \( (A \lor B)^{Sh} \equiv \forall u, v \exists x, y (A_{sh}(u, x) \lor B_{sh}(v, y)) \)

**Sh4**  \( (\forall z A)^{Sh} \equiv \forall z, u \exists x A_{sh}(z, u, x) \)

**Sh5**  \( (A \rightarrow B)^{Sh} \equiv \forall f, v \exists u, y (A_{sh}(u, f(u)) \rightarrow B_{sh}(v, y)) \)

**Sh6**  \( (\exists z A)^{Sh} \equiv \forall U \exists z, f A_{sh}(z, U(z, f), f(U(z, f))) \)

**Sh7**  \( (A \land B)^{Sh} \equiv \forall n, u, v \exists x, y (n=0 \rightarrow A_{sh}(u, x)) \land (n \neq 0 \rightarrow B_{sh}(v, y)) \)

\[ \leftrightarrow \forall u, v \exists x, y (A_{sh}(u, x) \land B_{sh}(v, y)). \]
Sh extracts from a given proof $p$

$$p \models \forall x \exists y \ A_{qf}(x, y)$$

explicit effective functionals $t$ realizing $A^{Sh}$, i.e.

$$\forall x \ A_{qf}(x, t(x)).$$

Gödel, Kreisel, Spector, Parsons, Feferman, K., . . .: Numerous optimal extraction theorems for systems from poly-time arithmetic up to full analysis with dependent choice (Spector, based on bar recursion).

Solution for (IPP)$^{Sh}$ (P. Oliva 2006): Based on a weak finitary bar recursion (highly nontrivial).
This time: Correct and in fact optimal complexity!
Monotone $Sh$ extracts $\Phi^*$ such that

$$\exists Y \left( \Phi^* \succeq Y \land \forall x A_{Sh}(x, Y(x)) \right),$$
Monotone functional interpretation MFI (K.1996)

Monotone $Sh$ extracts $\Phi^*$ such that

$$\exists Y \left( \Phi^* \gtrsim Y \land \forall x \ A_{Sh}(x, Y(x)) \right),$$

where $\gtrsim$ is some suitable notion of being a ‘bound’ that applies to higher order function spaces (W.A. Howard)

$$\left\{ \begin{array}{l}
  x^* \gtrsim_{IN} x : \equiv x^* \geq x, \\
  x^* \gtrsim_{\rho \to \tau} x : \equiv \forall y^*, y(y^* \gtrsim_{\rho} y \rightarrow x^*(y^*) \gtrsim_{\tau} x(y)).
\end{array} \right.$$
Monotone interpretation of PCM

Let \((a_n)\) be a nondecreasing sequence in \([0, 1]\) and hence Cauchy (PCM). The monotone G-interpretation asks for a bound \(\Phi\) such that for all \(k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N}\)

\[\exists N \leq \Phi(k, g) \forall n, m \in [N, N + g(N)](|a_n - a_m| \leq 2^{-k}).\]

**Solution:** Take \(\Phi(k, g) := \tilde{g}(2^k - 1)(0)\), where \(\tilde{g}(n) := n + g(n)\).
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**Corollary**

(T. Tao’s finite convergence principle, 2007)

\[\forall k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N} \exists M \in \mathbb{N} \forall 0 \leq a_0 \leq \ldots \leq a_M \leq 1 \exists N \in \mathbb{N} (N + g(N) \leq M \land \forall n, m \in [N, N + g(N)](|a_n - a_m| \leq 2^{-k})).\]

In fact, we take \(M := \tilde{g}(2^k)(0).\)
Monotone interpretation of IPP

The monotone functional interpretation of IPP yields a version of Tao's 'finitary' IPP.
The monotone functional interpretation of IPP yields a *version* of Tao's 'finitary' IPP.

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‘it is common to make a distinction between “hard”, “quantitative”, or “finitary” analysis on the one hand, and “soft”, “qualitative”, or “infinitary” analysis on the other hand.’ ...‘It is fairly well known that the results obtained by hard and soft analysis resp. can be connected to each other by various “correspondence principles” or “compactness principles”. It is however my belief that the relationship between the two types of analysis is much deeper.’ ...‘There are rigorous results from proof theory which can allow one to automatically convert certain types of qualitative arguments into quantitative ones...’

(T. Tao: Soft analysis, hard analysis, and the finite convergence principle, 2007)
General logical metatheorems

Metatheorems on the extractability of uniform bounds in analysis: ‘if a sentence has a certain logical form and can be proved in $\mathcal{T}$, then the following strengthened version holds: ...’.

- Logical form: meaningful for restricted languages.
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- Logical form: meaningful for **restricted languages**.
- **Concrete Polish ($P$) and compact ($K$) metric spaces** are represented via $\mathbb{N}^\mathbb{N}$ and $2^\mathbb{N}$. Macros ‘$\forall x \in P, y \in K$’.
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- **concrete Polish $(P)$ and compact $(K)$ metric spaces** are represented via $\mathbb{N}^\mathbb{N}$ and $2^\mathbb{N}$. Macros ‘$\forall x \in P, y \in K$’.
- **Many abstract types of metric structures can be added as atoms**: metric, hyperbolic, CAT(0), $\delta$-hyperbolic, normed, uniformly convex, Hilbert, … spaces or $\mathbb{R}$-trees $X$: add **new base type $X$**, all **finite types over $\mathbb{N}, X$** and a new **constant $d_X$** representing $d$ etc.
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- **Many abstract types of metric structures can be added as atoms**: metric, hyperbolic, CAT(0), $\delta$-hyperbolic, normed, uniformly convex, Hilbert, ... spaces or $\mathbb{R}$-trees $X$ : add **new base type** $X$, all **finite types over** $\mathbb{N}, X$ and a new **constant** $d_X$ representing $d$ etc. **Crucial** for uniformity of bounds: **no separability assumptions**!
A formal system for analysis

**Types:** (i) $\mathbb{N}, X$ are types, (ii) with $\rho, \tau$ also $\rho \rightarrow \tau$ is a type.

Functionals of type $\rho \rightarrow \tau$ map type-$\rho$ objects to type-$\tau$ objects.
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$\text{PA}^\omega, X$ is the extension of Peano Arithmetic to all types.

$\text{A}^\omega, X := \text{PA}^\omega, X + \text{DC}$, where

**DC:** axiom of dependent choice for all types

Implies **full comprehension** for numbers (higher order arithmetic).
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Implies *full comprehension* for numbers (higher order arithmetic).

$A^{\omega}[X, d, \ldots]$ results by adding constants $d_X, \ldots$ with axioms expressing that $(X, d, \ldots)$ is a nonempty metric, hyperbolic, normed, Hilbert

\ldots space.
A warning concerning equality

**Extensionality rule** *(only!)*:

\[
\frac{s =_\rho t}{\tau r(s) = \tau r(t)},
\]

where only \( x =_\IN y \) primitive equality predicate but for \( \rho \rightarrow \tau \)

\[
s^X =_X t^X \equiv d_X(x, y) =_{\IR} 0_{\IR},
\]

\[
s =_{\rho \rightarrow \tau} t \equiv \forall v^{\rho}(s(v) =_\tau t(v)).
\]
A novel form of majorization

\[ y, x \text{ functionals of types } \rho, \hat{\rho} := \rho[\mathbb{N}/X] \text{ and } a^X \text{ of type } X: \]

\[
x^\mathbb{N} \geq^a y^\mathbb{N} :\equiv x \geq y
\]

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x^\mathbb{N} \geq^a_X y^X :\equiv x \geq d(y, a).
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$y, x$ functionals of types $\rho, \hat{\rho} := \rho[\mathbb{N}/X]$ and $a^X$ of type $X$:

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    x^\mathbb{N} \succeq_{\mathbb{N}}^a y^\mathbb{N} & \equiv x \geq y \\
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For complex types $\rho \rightarrow \tau$ this is extended in a hereditary fashion.
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For complex types \( \rho \rightarrow \tau \) this is extended in a hereditary fashion.

**Example:**

\[ f^* \succ^a_{X \rightarrow X} f \equiv \forall n \in \mathbb{N}, x \in X[n \geq d(a, x) \rightarrow f^*(n) \geq d(a, f(x))]. \]
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\( y, x \) functionals of types \( \rho, \hat{\rho} := \rho[\mathbb{N}/X] \) and \( a^X \) of type \( X \):

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\( f : \mathbb{X} \rightarrow \mathbb{X} \) is nonexpansive (n.e.) if \( d(f(x), f(y)) \leq d(x, y) \).

Then \( \lambda n.n + b \succeq_a^{\mathbb{X} \rightarrow \mathbb{X}} f \), if \( d(a, f(a)) \leq b \).
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$f : X \rightarrow X$ is nonexpansive (n.e.) if $d(f(x), f(y)) \leq d(x, y)$.

Then $\lambda n.n + b \succeq_{X \rightarrow X}^a f$, if $d(a, f(a)) \leq b$.

**Normed linear case:** $a := 0_X$. 

Recent Applications of Proof Theory in Fixed Point and Ergodic Theory
Theorem (Gerhardy/K., Trans. Amer. Math. Soc. 2008)

Let $P$, $K$ be Polish resp. compact metric spaces, $A_\exists$ $\exists$-formula, 
$\tau$ ‘small’ (e.g. $X, \mathbb{N} \to X$ or $X \to X$).
If $A^\omega[X, d, \ldots]$ proves

$$\forall x \in P \forall y \in K \forall z^\tau \exists v^\mathbb{N} A_\exists(x, y, z, v),$$

then one can extract a computable $\Phi : \mathbb{N}^\mathbb{N} \times (\mathbb{N}^\mathbb{N}) \to \mathbb{N}$ s.t. for all metric (hyperbolic, \ldots) spaces, all representatives $r_x \in \mathbb{N}^\mathbb{N}$ of $x \in P$ and all $z^\tau$ and $z^* \in \mathbb{N}^{(\mathbb{N})}$ s.t. $\exists a \in X(z^* \gtrsim^\tau a z)$:

$$\forall y \in K \exists v \leq \Phi(r_x, z^*) A_\exists(x, y, z, v).$$

For the bounded cases: K. Trans. AMS 2005.
As special case of \textbf{general logical metatheorems} due to Gerhardy/K. (Trans. Amer. Math. Soc. 2008) one has:

\textbf{Corollary (Gerhardy/K., TAMS 2008)}

If $A^\omega [X, d, \ldots]$ proves

$$\forall x \in P \forall y \in K \forall z \in X \forall f : X \to X \left( f \text{ n.e. } \to \exists v \in \mathbb{N} A_\exists \right),$$

then one can extract a \textbf{computable functional} $\Phi : \mathbb{N}^\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ s.t.

for all $x \in P, b \in \mathbb{N}$

$$\forall y \in K \forall z \in X \forall f : X \to X \left( f \text{ n.e. } \land d_X(z, f(z)) \leq b \to \exists v \leq \Phi(r_x, b) A_\exists \right)$$

holds in \textbf{all metric (hyperbolic, \ldots) spaces} $(X, d)$. 

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holds in all metric (hyperbolic, \ldots) spaces $(X, d)$.

Also for **normed** and **(pre-)Hilbert** spaces with $b \geq \|z\|$. 
Proof Mining in Ergodic Theory

Let $X$ be a **Hilbert space**, $f : X \to X$ linear and nonexpansive. 

$$A_n(x) := \frac{1}{n+1} S_n(x), \text{ where } S_n(x) := \sum_{i=0}^{n} f^i(x) \quad (n \geq 0).$$
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**Theorem (von Neumann Mean Ergodic Theorem)**

For every $x \in X$, the sequence $(A_n(x))_n$ converges.
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For every $x \in X$, the sequence $(A_n(x))_n$ converges.

Avigad/Gerhardy/Towsner (TAMS to appear):
in general **no computable rate of convergence**.
Proof Mining in Ergodic Theory

Let $X$ be a **Hilbert space**, $f : X \to X$ **linear and nonexpansive**.

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**Theorem (Garrett Birkhoff 1939)**

Mean Ergodic Theorem holds for uniformly convex Banach spaces.
Based on logical metatheorem discussed above:

Theorem (K./Leuştean, to appear in Ergodic Theor. Dynam. Syst.)

Assume that $X$ is a uniformly convex Banach space, $\eta$ is a modulus of uniform convexity and $f : X \to X$ is a nonexpansive linear operator. Let $b > 0$. Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \leq \Phi(\varepsilon, g, b, \eta) \forall i, j \in [n, n + g(n)] (\|A_i(x) - A_j(x)\| < \varepsilon),$$
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where

$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^K(0), \text{ with }$$

$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16} \eta \left( \frac{\varepsilon}{8b} \right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$

$$h, \tilde{h} : \mathbb{N} \to \mathbb{N}, \quad h(n) := 2(Mn + g(Mn)), \quad \tilde{h}(n) := \max_{i \leq n} h(i).$$
Corollary (K./Leuştean 2008)

Let $X$ be a Hilbert space and $f : X \to X$ be a nonexpansive linear operator. Let $b > 0$. Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

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where $\Phi$ is defined as above, but with $K := \left\lceil \frac{512b^2}{\varepsilon^2} \right\rceil$. 

Discussion: The Hilbert space case has been treated (again based on our metatheorem) prior by Avigad-Gerhardy-Towsner (TAMS to appear). However, the bound obtained by Avigad et al. is less good and matches our bound only in the special case of isometric $f$. 

Recent Applications of Proof Theory in Fixed Point and Ergodic Theory
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‘We shall establish Theorem 1.6 by “finitary ergodic theory” techniques, reminiscent of those used in [Green-Tao]...’ ‘The main advantage of working in the finitary setting ... is that the underlying dynamical system becomes extremely explicit’...‘In proof theory, this finitisation is known as Gödel functional interpretation...which is also closely related to the Kreisel no-counterexample interpretation’ (T. Tao: Norm convergence of multiple ergodic averages for commuting transformations, Ergodic Theor. and Dynam. Syst. 28, 2008)
\( \mathcal{A}^\omega [X, \langle \cdot, \cdot \rangle] \) does not have nontrivial comprehension over \( X \)-type objects but proves (using countable choice for \( X \)-objects) \textbf{schematically} the existence of unique best approximations (resp. orthogonal projections), for linear functionals \( L : X \to \mathbb{R} \) with definable graph, the Riesz representation theorem, and the weak compactness of \( B_1(0) \) (here only countable choice for arithmetical formulas needed and restricted induction).
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A theorem of F.E. Browder

Using projection to the set of all fixed points of a nonexpansive mapping $U : X \rightarrow X$ with $U|_{B_1(0)} : B_1(0) \rightarrow B_1(0)$ ($X$ Hilbert space) and weak compactness, Browder showed in 1967:

Theorem: For $n \in \mathbb{N}$, $v_0 \in B_1(0)$ let $u_n$ be the unique fixed point of the contraction $U_n(x) := (1 - 1/n)U(x) - 1/n v_0$. Then $(u_n)$ converges towards the fixed point of $U$ that is closest to $v_0$.

Corollary by Metatheorem: There is a functional $\Phi(k, g)$ (definable by primitive recursion and bar recursion of lowest type) such that

$$\forall k \in \mathbb{N} \forall g : \mathbb{N} \rightarrow \mathbb{N} \exists n \leq \Phi(k, g) \forall i, j \in [n; n + g(n)] (\|u_i - u_j\| < 2^{-k})$$

Note that $\Phi$ does not depend on $U$, $v_0$ or $X$!
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Kirk’s theorem for asymptotic contractions

**Definition (Kirk JMAA03)**

$\left( X, d \right)$ metric space. $f : X \to X$ is an **asymptotic contraction** with moduli $\Phi, \Phi_n : [0, \infty) \to [0, \infty)$ if $\Phi, \Phi_n$ are continuous, $\Phi(s) < s$ for all $s > 0$ and

$$\forall n \in \mathbb{N} \forall x, y \in X \left( d(f^n(x), f^n(y)) \leq \Phi_n(d(x, y)) \right),$$

and $\Phi_n \to \Phi$ uniformly on the range of $d$. 

Theorem (Kirk JMAA03)

$\left( X, d \right)$ complete metric space, $f : X \to X$ continuous asymptotic contraction with some orbit bounded. Then $f$ has a unique fixed point $p \in X$ and ($f^n(x_0)$) converges to $p$ for each $x_0 \in X$.

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By proof mining P. Gerhardy (JMAA 2006, communicated by Kirk) obtained an effective rate of proximity $\Phi$ in appropriate moduli with elementary proof such that for the fixed point $p$

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Using the uniformity of Gerhardy's result, E.M. Briseid (JMAA 2007) constructed an effective full rate of convergence. As a consequence of his analysis E.M. Briseid showed that the $(f^n(x_0))$ is redundant to assume: rate of convergence using only $b \geq d(x, f(x))$ (Fixed Point Theory 2007, Int. J. Math. Stat. 2010).

E.M. Briseid showed that for bounded metric spaces the existence of a $x_0$-uniform rate of convergence implies that $f$ is asymptotically. Also improvements of recent uniformity results of Reich, Zaslavski et al. (2007). contractive (JMAA 2007).
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Further applications of proof theory to analysis


Ulrich Kohlenbach presents an applied form of proof theory that has led in recent years to new results in number theory, approximation theory, nonlinear analysis, geodesic geometry and ergodic theory (among others). This applied approach is based on logical transformations (so-called proof interpretations) and concerns the extraction of effective data (such as bounds) from prima facie ineffective proofs as well as new qualitative results such as independence of solutions from certain parameters, generalizations of proofs by elimination of premises.

The book first develops the necessary logical machinery emphasizing novel forms of Gödel’s famous functional („Dialectica”) interpretation. It then establishes general logical metatheorems that connect these techniques with concrete mathematics. Finally, two extended case studies (one in approximation theory and one in fixed point theory) show in detail how this machinery can be applied to concrete proofs in different areas of mathematics.