Existence of Periodic Solutions of the Solow Equation with Periodic Labor Rate Growth

João Teixeira
Departamento de Matemática, Instituto Superior Técnico, Lisboa, Portugal

August 1, 2008
1. The Solow Equation

2. Discretization of the Solow equation problem

3. Discrete fixed point problem

4. Iterative solution of the hyperfinite problem

5. Existence of a periodic solution
The Solow equation periodic solution problem

The periodic solution problem for the Solow equation is:

\[
\begin{align*}
\frac{du}{dt} &= s f(k) - (\delta + n(t)) k & \text{in } \mathbb{R} \\
\quad k(t) &= k(t + T) & \text{for all } t \in \mathbb{R}
\end{align*}
\]  

(1)
The periodic solution problem for the Solow equation is:

\[
\begin{cases}
\frac{du}{dt} = s f(k) - (\delta + n(t)) k \\ k(t) = k(t + T)
\end{cases} \quad \text{in } \mathbb{R}
\]

for all \( t \in \mathbb{R} \)

- \([0, \infty) \ni k \mapsto f(k) \in [0, \infty)\) is the production function.
- \( s > 0 \) is the savings rate.
- \( \delta > 0 \) is the capital depreciation.
- \( \mathbb{R} \ni t \mapsto n(t) \in \mathbb{R} \) is the labor force growth rate.
The Solow equation periodic solution problem

The periodic solution problem for the Solow equation is:

\[
\begin{cases}
\frac{du}{dt} = s f(k) - (\delta + n(t)) k \\
k(t) = k(t + T)
\end{cases}
\]

in \( \mathbb{R} \)

for all \( t \in \mathbb{R} \)

- \([0, \infty) \ni k \mapsto f(k) \in [0, \infty)\) is the production function.
- \(s > 0\) is the savings rate.
- \(\delta > 0\) is the capital depreciation.
- \(\mathbb{R} \ni t \mapsto n(t) \in \mathbb{R}\) is the labor force growth rate.

A solution is a \(C^1\) function \(\mathbb{R} \ni t \mapsto k(t) \in \mathbb{R}\), and it represents the output of capital per worker.
Conditions on $f(k)$ and $n(t)$

The production function, $f$, satisfies:

1. $f$ is strictly increasing.
2. $f$ is locally Lipschitz continuous on $(0, \infty)$.
3. For every $m > 0$, there exists a unique $k^\bullet > 0$ satisfying:
   - $sf(k^\bullet) - mk^\bullet = 0$
   - $sf(k) - mk > 0$ for $0 < k < k^\bullet$
   - $sf(k) - mk < 0$ for $k > k^\bullet$

The labor force growth rate, $n(t)$, satisfies:

1. $n$ is continuous.
2. For some $T > 0$, $n(t) = n(t + T)$ for all $t \in \mathbb{R}$.
3. $n$ is strictly positive.
Conditions on $f(k)$ and $n(t)$

The production function, $f$, satisfies:

1. $f$ is strictly increasing.
2. $f$ is locally Lipschitz continuous on $(0, \infty)$.
3. For every $m > 0$, there exists a unique $k^* > 0$ satisfying:
   - $sf(k^*) - mk^* = 0$
   - $sf(k) - mk > 0$ for $0 < k < k^*$
   - $sf(k) - mk < 0$ for $k > k^*$

The labor force growth rate, $n(t)$, satisfies:

1. $n$ is continuous.
2. For some $T > 0$, $n(t) = n(t + T)$ for all $t \in \mathbb{R}$.
3. $n$ is strictly positive.
If $n(t) = n_0 > 0$, then there exists $k^\bullet > 0$ such that:

$$s f(k^\bullet) - (\delta + n_0)k^\bullet = 0$$

Then $k(t) = k^\bullet$ is a solution of problem (1).
If \( n(t) = n_0 > 0 \), then there exists \( k^\bullet > 0 \) such that:

\[
 s f(k^\bullet) - (\delta + n_0)k^\bullet = 0
\]

Then \( k(t) = k^\bullet \) is a solution of problem (1).

We will study the case where \( n(t) \) is strictly \( T \)-periodic. Let \( I = [0, T] \) and define:

\[
\underline{n} = \min_{t \in I} n(t) \quad \bar{n} = \max_{t \in I} n(t)
\]

Then \( 0 < \underline{n} < \bar{n} \).
Discretization of time and gridfunctions

- Discretization of $[0, T]$, with $T$ identified with 0.

$$I_N = \left\{0, \Delta t, 2\Delta t, \ldots, (N - 1)\Delta t, T\right\}$$

$$= \Delta t \left(\mathbb{Z} \text{ mod } N\right);$$

with $N \in \mathbb{N}_1$, and $\Delta t = \frac{T}{N}$. 
Discretization of time and gridfunctions

- Discretization of $[0, T]$, with $T$ identified with 0.

\[ I_N = \left\{ 0, \Delta t, 2\Delta t, \ldots, (N-1)\Delta t, T \right\} \]

\[ = \Delta t (\mathbb{Z} \text{ mod } N); \]

with $N \in \mathbb{N}_1$, and $\Delta t = \frac{T}{N}$.

Given any $t = m\Delta t$ and $\tau = n\Delta t$ in $I_N$, let:

\[ t + \tau = \left( (m + n) \text{ mod } N \right) \Delta t \]
Discretization of time and gridfunctions

- Discretization of $[0, T]$, with $T$ identified with 0.

  \[ I_N = \left\{ 0, \Delta t, 2\Delta t, \ldots, (N - 1)\Delta t, T \right\} = \Delta t (\mathbb{Z} \mod N); \]

  with $N \in \mathbb{N}_1$, and $\Delta t = \frac{T}{N}$.

  Given any $t = m\Delta t$ and $\tau = n\Delta t$ in $I_N$, let:

  \[ t + \tau = \left( (m + n) \mod N \right) \Delta t \]

- Gridfunctions

  \[ K : I_N \rightarrow \mathbb{R} \]
Discretization of time and gridfunctions

- Discretization of $[0, T]$, with $T$ identified with 0.

$$I_N = \left\{ 0, \Delta t, 2\Delta t, \ldots, (N - 1)\Delta t, T \right\} = \Delta t \left( \mathbb{Z} \mod N \right);$$

with $N \in \mathbb{N}_1$, and $\Delta t = \frac{T}{N}$.

Given any $t = m\Delta t$ and $\tau = n\Delta t$ in $I_N$, let:

$$t + \tau = \left( (m + n) \mod N \right) \Delta t$$

- Gridfunctions

$$K : I_N \to \mathbb{R}$$

- This take care of the periodicity condition on the discretized problem.
An implicit Euler scheme

- The Euler “iterates” for the differential equation are:

\[ K(t + \Delta t) = K(t) + \Delta t \left( s f(K(t)) - (\delta + n(t))K(t) \right) \quad t \in I_N \tag{2} \]
An implicit Euler scheme

- The Euler “iterates” for the differential equation are:

\[
K(t + \Delta t) = K(t) + \Delta t \left( s f(K(t)) - (\delta + n(t))K(t) \right) \quad t \in I_N
\]

(2)

- In fact, this is an implicit scheme. For each \( N \in \mathbb{N}_1 \), there is a system of \( N \) equations in the unknowns \( K(t), t \in I_N \).
Let $k^\bullet$, $\bar{k}^\bullet$ be such that:

$$s f(k^\bullet) - (\delta + \bar{n})k^\bullet = 0$$ and $$s f(\bar{k}^\bullet) - (\delta + n)\bar{k}^\bullet = 0$$
Let $k^\bullet, \bar{k}^\bullet$ be such that:

$$sf(k^\bullet) - (\delta + \bar{n})k^\bullet = 0 \quad \text{and} \quad sf(\bar{k}^\bullet) - (\delta + n)\bar{k}^\bullet = 0$$

\(\bar{n} > n\) and condition (3) on \(f\) imply:

$$k^\bullet < \bar{k}^\bullet$$
Let $k^\bullet$, $\bar{k}^\bullet$ be such that:

$$sf(k^\bullet) - (\delta + \bar{n})k^\bullet = 0 \quad \text{and} \quad sf(\bar{k}^\bullet) - (\delta + n)\bar{k}^\bullet = 0$$

$\bar{n} > n$ and condition (3) on $f$ imply:

$$k^\bullet < \bar{k}^\bullet$$

The iteration map, $\Phi : \mathbb{R}^l \rightarrow \mathbb{R}^l$ is given by:

$$\Phi(K)(t) = \begin{cases} 
K(t) + \Delta t \left( sf(K(t)) - (\delta + n(t))K(t) \right) & \text{if } k^\bullet \leq K(t) \leq \bar{k}^\bullet \\
\frac{\bar{k}^\bullet + k^\bullet}{2} & \text{otherwise}
\end{cases}$$

(3)
Discrete fixed point problem

By construction of $\Phi$

$$k^* \leq \Phi(K)(t) \leq \bar{k}^* \quad \text{for all } t \in I_N$$
By construction of $\Phi$

\[ k^* \leq \Phi(K)(t) \leq \bar{k}^* \quad \text{for all } t \in l_N \]

Consider the shift map, given by $\sigma(K)(t) = K(t + \Delta t)$, for all $t \in l_N$. $K$ is a fixed point of $\sigma^{-1}\Phi$ iff $\sigma(K) = \Phi(K)$.
The Solow Equation
Discretization of the Solow equation problem
Discrete fixed point problem
Iterative solution of the hyperfinite problem
Existence of a periodic solution

Discrete fixed point problem

By construction of $\Phi$

$$k^\bullet \leq \Phi(K)(t) \leq \bar{k}^\bullet \quad \text{for all } t \in l_N$$

Consider the shift map, given by $\sigma(K)(t) = K(t + \Delta t)$, for all $t \in l_N$. $K$ is a fixed point of $\sigma^{-1}\Phi$ iff $\sigma(K) = \Phi(K)$.

**Lemma**

If $K$ is a fixed point of $\sigma^{-1}\Phi$ (on $\mathbb{R}^{l_N}$), such that

$$k^\bullet \leq K(t) \leq \bar{k}^\bullet,$$

then it satisfies the Euler implicit scheme (2)
Existence of a fixed point

Lemma

\( \sigma^{-1} \Phi \) has a fixed point.
Existence of a fixed point

Lemma

\( \sigma^{-1}\Phi \) has a fixed point.

Proof:
Lemma

\( \sigma^{-1} \Phi \) has a fixed point.

Proof:

- Let \( \|K\| = \max_{t \in I_N} |K(t)| \).
Existence of a fixed point

Lemma

\( \sigma^{-1}\Phi \) has a fixed point.

Proof:

- Let \( \|K\| = \max_{t \in I_N} |K(t)| \).
- Then \( \|\sigma^{-1}\Phi(K)\| = \|\Phi(K)\| \leq \bar{k}^* \) (by construction of \( \Phi \)).
Existence of a fixed point

Lemma

\( \sigma^{-1} \Phi \) has a fixed point.

Proof:

- Let \( \| K \| = \max_{t \in I_N} |K(t)| \).
- Then \( \| \sigma^{-1} \Phi(K) \| = \| \Phi(K) \| \leq \bar{k}^\bullet \) (by construction of \( \Phi \)).
- By the Brouwer fixed point theorem, there is a \( K \), with \( \| K \| \leq \bar{k}^\bullet \), such that \( \sigma^{-1} \Phi(K) = K \) \( \square \)
Main hyperfinite estimate

We now work in $\langle V(\mathbb{R}), \ast V(\mathbb{R}), \ast \rangle$

From now on, fix some $N \in \ast \mathbb{N} \setminus \mathbb{N}$. 

Lemma

For all $t \in I_N$, if $k \bullet \leq K(t) \leq \bar{k} \bullet$ then $k \bullet \leq \Phi(K(t)) \leq \bar{k} \bullet$.

Proof (for $K(t) \leq \bar{k} \bullet$, other inequality is similar):

Fix $t \in T_N$, and let $\epsilon > 0$ such that $K(t) = \bar{k} \bullet - \epsilon$.

WLOG, $\epsilon \approx 0$; and so $\Phi(K(t)) \leq K(t) + \Delta t \epsilon (\delta + n + sL)$, where $L$ is the Lipshitz constant of $f$ on $[k \bullet, \bar{k} \bullet]$.

Then $\Phi(K(t)) \leq \bar{k} \bullet + \epsilon (\frac{1}{\epsilon} - 1 + \Delta t (\delta + n + sL)) \leq \bar{k} \bullet$. □
Main hyperfinite estimate

We now work in $\langle V(\mathbb{R}), \ast V(\mathbb{R}), \ast \rangle$

From now on, fix some $N \in \ast \mathbb{N} \setminus \mathbb{N}$.

Lemma

For all $t \in l_N$, if $k^\bullet \leq K(t) \leq \bar{k}^\bullet$ then $k^\bullet \leq \Phi(K)(t) \leq \bar{k}^\bullet$. 
Main hyperfinite estimate

We now work in \( \langle V(\mathbb{R}), \ast V(\mathbb{R}), \ast \rangle \)

From now on, fix some \( N \in \ast \mathbb{N} \setminus \mathbb{N} \).

**Lemma**

For all \( t \in I_N \), if \( k^\ast \leq K(t) \leq \bar{k}^\ast \) then \( k^\ast \leq \Phi(K)(t) \leq \bar{k}^\ast \).

Proof (for \( K(t) \leq \bar{k}^\ast \), other inequality is similar):
We now work in $\langle V(\mathbb{R}), *V(\mathbb{R}), * \rangle$
From now on, fix some $N \in \ast \mathbb{N} \setminus \mathbb{N}$.

Lemma

For all $t \in I_N$, if $k^* \leq K(t) \leq \bar{k}^*$ then $k^* \leq \Phi(K)(t) \leq \bar{k}^*$.

Proof (for $K(t) \leq \bar{k}^*$, other inequality is similar):
- Fix $t \in T_N$, and let $\epsilon > 0$ such that $K(t) = \bar{k}^* - \epsilon$. 
Main hyperfinite estimate

We now work in $\langle V(\mathbb{R}), *V(\mathbb{R}), * \rangle$
From now on, fix some $N \in *\mathbb{N} \setminus \mathbb{N}$.

**Lemma**

For all $t \in I_N$, if $k^\bullet \leq K(t) \leq \bar{k}^\bullet$ then $k^\bullet \leq \Phi(K)(t) \leq \bar{k}^\bullet$.

Proof (for $K(t) \leq \bar{k}^\bullet$, other inequality is similar):
- Fix $t \in T_N$, and let $\epsilon > 0$ such that $K(t) = \bar{k}^\bullet - \epsilon$.
- WLOG, $\epsilon \approx 0$;
Main hyperfinite estimate

We now work in \( \langle V(\mathbb{R}), \ast V(\mathbb{R}), \ast \rangle \).

From now on, fix some \( N \in *\mathbb{N} \setminus \mathbb{N} \).

Lemma

For all \( t \in I_N \), if \( k^\bullet \leq K(t) \leq \bar{k}^\bullet \) then \( k^\bullet \leq \Phi(K)(t) \leq \bar{k}^\bullet \).

Proof (for \( K(t) \leq \bar{k}^\bullet \), other inequality is similar):

- Fix \( t \in T_N \), and let \( \epsilon > 0 \) such that \( K(t) = \bar{k}^\bullet - \epsilon \).
- WLOG, \( \epsilon \approx 0 \); and so

\[
\Phi(K)(t) \leq K(t) + \Delta t \epsilon (\delta + \bar{n} + sL),
\]

where \( L \) is the Lipshitz constant of \( f \) on \( [k^\bullet, \bar{k}^\bullet] \).
Main hyperfinite estimate

We now work in $\langle V(\mathbb{R}), *V(\mathbb{R}), * \rangle$
From now on, fix some $N \in \ast \mathbb{N} \setminus \mathbb{N}$.

Lemma

For all $t \in I_N$, if $k^* \leq K(t) \leq \bar{k}^*$ then $k^* \leq \Phi(K)(t) \leq \bar{k}^*$.

Proof (for $K(t) \leq \bar{k}^*$, other inequality is similar):

1. Fix $t \in T_N$, and let $\epsilon > 0$ such that $K(t) = \bar{k}^* - \epsilon$.
2. WLOG, $\epsilon \approx 0$; and so

$$\Phi(K)(t) \leq K(t) + \Delta t \epsilon (\delta + \bar{n} + sL),$$

where $L$ is the Lipshitz constant of $f$ on $[k^*, \bar{k}^*]$.
3. Then $\Phi(K)(t) \leq \bar{k}^* + \epsilon(-1 + \Delta t (\delta + \bar{n} + sL)) \leq \bar{k}^*$. □
Lemma

The implicit Euler scheme (2) has a solution $K : l_N \rightarrow \ast\mathbb{R}$ that satisfies $k^\bullet \leq K(t) \leq \bar{k}^\bullet$, for all $t \in l_N$. 

Proof:
We established that $\sigma^{-1}\Phi$ has a fixed point, $K$.

Assume $K(t)$ is outside $\ast[k^\bullet, \bar{k}^\bullet]$, for some $t \in l_N$.

Then by construction of $\Phi$, $K(t + \Delta t) \in \ast[k^\bullet, \bar{k}^\bullet]$.

By the preceding Lemma: $K(t + 2\Delta t), K(t + 3\Delta t), \ldots, K(t + N\Delta t) = K(t)$ is in $\ast[k^\bullet, \bar{k}^\bullet]$.

□
Solution of the implicit Euler scheme

Lemma

The implicit Euler scheme \((2)\) has a solution \(K : I_N \rightarrow \mathbb{R}^\star\) that satisfies \(k^\star \leq K(t) \leq \bar{k}^\star\), for all \(t \in I_N\).

Proof:

- We established that \(\sigma^{-1} \Phi\) has a fixed point, \(K\)
Solution of the implicit Euler scheme

Lemma

The implicit Euler scheme (2) has a solution $K : l_N \rightarrow ^*\mathbb{R}$ that satisfies $k^\bullet \leq K(t) \leq \bar{k}^\bullet$, for all $t \in l_N$.

Proof:

- We established that $\sigma^{-1}\Phi$ has a fixed point, $K$
- Assume $K(t)$ is outside $^[k^\bullet, \bar{k}^\bullet]$, for some $t \in l_N$. 

João Teixeira  Periodic Solutions of the Solow Equation
Solution of the implicit Euler scheme

Lemma

The implicit Euler scheme (2) has a solution \( K : I_N \to ^\ast \mathbb{R} \) that satisfies \( \underline{k} \leq K(t) \leq \overline{k} \), for all \( t \in I_N \).

Proof:

1. We established that \( \sigma^{-1} \Phi \) has a fixed point, \( K \)
2. Assume \( K(t) \) is outside \( ^*\left[\underline{k}, \overline{k} \right] \), for some \( t \in I_N \).
3. Then by construction of \( \Phi \), \( K(t + \Delta t) \in ^*\left[\underline{k}, \overline{k} \right] \).
The implicit Euler scheme (2) has a solution $K : l_N \rightarrow \ast \mathbb{R}$ that satisfies $\underline{k} \leq K(t) \leq \bar{k}$, for all $t \in l_N$.

Proof:

- We established that $\sigma^{-1}\Phi$ has a fixed point, $K$
- Assume $K(t)$ is outside $\ast[\underline{k}, \bar{k}]$, for some $t \in l_N$.
- Then by construction of $\Phi$, $K(t + \Delta t) \in \ast[\underline{k}, \bar{k}]$.
- By the preceding Lemma:

$$K(t + 2\Delta t), K(t + 3\Delta t), \ldots, K(t + N\Delta t) = K(t)$$

is in $\ast[\underline{k}, \bar{k}]$!
The periodic extension of $K$ is Euler-recursive

Let $\bar{K}$ be the periodic extension of $K$ to the whole set of gridpoints, $\Delta t(\ast\mathbb{Z})$:

$$\bar{K}(n\Delta t) = K((n \mod N)\Delta t), \quad \text{for any } n \in \ast\mathbb{Z}. $$
The periodic extension of $K$ is Euler-recursive

Let $\bar{K}$ be the periodic extension of $K$ to the whole set of gridpoints, $\Delta t(*\mathbb{Z})$:

$$\bar{K}(n\Delta t) = K((n \mod N)\Delta t), \quad \text{for any } n \in *\mathbb{Z}.$$  

By the preceeding Lemma, it satisfies:

$$\bar{K}(t+\Delta t) = \bar{K}(t) + \Delta t\left(sf(\bar{K}(t)) - (\delta + n(t))\bar{K}(t)\right) \quad t \in I_N \quad (4)$$

where $\bar{K}(0) = K(0)$ is some (fixed) value in $*[k^\bullet, \bar{k}^\bullet]$. 
S-continuity of \( \bar{K} \)

**Lemma**

\( \bar{K} \) is S-continuous on \( \Delta t^* \mathbb{Z} \)

Proof: Follows from \( \bar{K}(t) \in \^k \[k^\cdot, \bar{k^\cdot}\] \)
Continuity of \( f \)
Boundedness of \( n \)
That
\[
|sf(\bar{K}(t)) - (\delta + n(t)) \bar{K}(t)|
\]
is bounded for all \( t \) (by a limited constant).
\( \square \)
S-continuity of $\bar{K}$

**Lemma**

$\bar{K}$ is $S$-continuous on $\Delta t^* \mathbb{Z}$

Proof: Follows from
- $\bar{K}(t) \in ^* [k^*, \bar{k}^*]$
- Continuity of $f$
- Boundedness of $n$

That

$$|sf(\bar{K}(t)) - (\delta + n(t))\bar{K}(t)|$$

is bounded for all $t$ (by a limited constant).  

□
A good candidate for solution is:

\[ k(\text{st } t) = \text{st } \tilde{K}(t) \quad \text{for all } t \in \Delta t^* \mathbb{Z}^n \]
A good candidate for solution is:

\[ k(\text{st } t) = \text{st } \bar{K}(t) \quad \text{for all } t \in \Delta t^* \mathbb{Z}^n \]

This function is well-defined on \( \mathbb{R} \) and is (globally) Lipschitz continuous.
A good candidate for solution is:

\[ k(\text{st } t) = \text{st } \bar{K}(t) \quad \text{for all } t \in \Delta t^* \mathbb{Z}^n \]

This function is well-defined on \( \mathbb{R} \) and is (globally) Lipshitz continuous. Furthermore, \( k \) is easily seen to be \( T \)-periodic.

**Theorem**

Let \( k \) be as above. Then \( k \in C^1(\mathbb{R}, \mathbb{R}) \) is a \( T \)-periodic solution of the Solow equation (1).

**Proof:** proceed with the remainder steps of the usual nonstandard proof of Peano's theorem. □
Existence theorem

A good candidate for solution is:

$$\text{k}(\text{st} \ t) = \text{st} \ \bar{K}(t) \quad \text{for all } t \in \Delta t^* \mathbb{Z}^n$$

This function is well-defined on $\mathbb{R}$ and is (globally) Lipschitz continuous. Furthermore, $\text{k}$ is easily seen to be $T$-periodic.

**Theorem**

*Let $\text{k}$ be as above. Then $\text{k} \in C^1(\mathbb{R}, \mathbb{R})$ is a $T$-periodic solution of the Solow equation (1).*

Proof: proceed with the remainder steps of the usual nonstandard proof of Peano’s theorem.