Horn-rule Completeness for Probabilistic Consequence Relations

6th August 2009
We work in the language of standard propositional calculus, with $SL$ being the set of sentences in the language. A consequence relation is a binary relation $\models$ on $SL$ representing some kind of inference.

A probability function on a propositional language is a function $w : SL \to [0, 1]$ such that

- if $\models a$ then $w(a) = 1$
- if $\models \neg(a \land b)$ then $w(a \lor b) = w(a) + w(b)$.

For each probability function $w$ and threshold $t \in [0, 1]$, a probabilistic consequence relation (PCR) $\models$ is defined by taking $a \models x$ whenever $w(a) = 0$ or $w(x|a) \geq t$. 

2
Preferential consequence relations ($\mathbf{P}$) are relations that obey

\[
\begin{align*}
  &a \not\sim a \quad \text{(REF)} \\
  \text{if } &a \not\sim x \text{ and } x \vdash y \text{ then } a \not\sim y \quad \text{(RW)} \\
  \text{if } &a \not\sim x \text{ and } a \not\vdash b \text{ then } b \not\sim x \quad \text{(LCE)} \\
  \text{if } &a \not\sim x \land y \text{ then } a \land x \not\sim y \quad \text{(VCM)} \\
  \text{if } &a \not\sim y \text{ and } b \not\sim y \text{ then } a \lor b \not\sim y \quad \text{(OR)} \\
  \text{if } &a \not\sim x \text{ and } a \not\sim y \text{ then } a \not\sim x \land y \quad \text{(AND)}
\end{align*}
\]

A representation theorem exists [3] showing that relations in $\mathbf{P}$ are those generated by stoppered preferential models.

Note that all the rules that characterise $\mathbf{P}$ are finite-premised Horn rules (with some side-conditions).
How do PCRs compare? Well... AND and OR fail. But if we replace them with these weakened versions

- if $a \land x \vdash y$ and $a \land \neg x \vdash y$ then $a \not\vdash y$ (WOR)
- if $a \not\vdash x$ and $a \land \neg y \vdash y$ then $a \not\vdash x \land y$ (WAND)

then PCRs do obey them. The system of REF, RW, LCE, VCM, WOR and WAND is called $\mathcal{O}$.

Note that the PCRs are not a superclass of $\mathbf{P}$. We have:

if $a \not\vdash x$ then either $a \land b \not\vdash x$ or $a \land \neg b \not\vdash x$ (NR)

which is sound for PCRs but not for $\mathbf{P}$.
Questions: Are there rules that will provide a representation theorem or completeness theorem for PCRs?

Hawthorne and Makinson [2] showed no such representation theorem exists in terms of Horn rules, and also no completeness theorem exists if infinite Horn rules are included. In fact for any probabilistically sound set of Horn rules there exists a relation in $\mathbf{P}$ that is not a PCR, but obeys all those rules.

But they left open the question of completeness for finite-premised Horn rules, suggesting $\mathcal{O}$ as a candidate.

(From here on all Horn rules are finite-premised.)
So when exactly will a Horn rule be probabilistically sound?

It turns out that if a Horn rule is sound for PCRs generated with threshold $s \in (0, 1]$ then it is sound for those generated using any threshold $t > s$.

A Horn rule is also always sound for PCRs generated from threshold 0. Therefore for a rule to be sound for all PCRs it suffices for it to be sound for PCRs generated with the thresholds $t = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$.

We can find several equivalent soundness conditions for Horn rules by first using results from linear algebra and then moving to a non-standard model of the reals.
For a Horn rule taking premises $a_1 \sim x_1, \ldots a_k \sim x_k$ to conclusion $a_0 \sim x_0$ the following are equivalent:

- the rule holds for all PCRs generated using thresholds $\frac{1}{q}$ ($0 < q \in \mathbb{N}$).

- for each such $q$ there exist $N_0, N_1, \ldots, N_k \in \mathbb{N}$ such that $N_0 > 0$ and

  $$N_0 (q \langle a_0 \land x_0 \rangle - \langle a_0 \rangle) \geq \sum_{i=1}^{k} N_i (q \langle a_i \land x_i \rangle - \langle a_i \rangle)$$

- for a fixed infinite natural $q \in *\mathbb{N}$ there exist $N_0, \ldots, N_k \in *\mathbb{N}$ such that $N_0 > 0$ and the following holds:
Consider the following table:

<table>
<thead>
<tr>
<th></th>
<th>$a_0 \land x_0$</th>
<th>$a_1 \land \neg x_1$</th>
<th>$\ldots$</th>
<th>$a_k \land \neg x_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_0$</td>
<td>$(1 - q)N_0$</td>
<td>$-N_1$</td>
<td>$\ldots$</td>
<td>$-N_k$</td>
</tr>
<tr>
<td>$\neg a_0$</td>
<td>0</td>
<td>$\neg a_1$</td>
<td>$\ldots$</td>
<td>$\neg a_k$</td>
</tr>
<tr>
<td>$0$</td>
<td>0</td>
<td>0</td>
<td>$\ldots$</td>
<td>0</td>
</tr>
<tr>
<td>$N_1$</td>
<td>$(q - 1)N_1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N_k$</td>
<td>$(q - 1)N_k$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Each cell contains a formula and an associated ‘cost’ in terms of $q$ and the $N_i$. Define a ‘path’ to be a choice of one cell from each column of the table, and the cost of a path to be the sum of its cells’ costs.

Our condition is: for any path whose cost exceeds 0, the sentences on the path must form a classically inconsistent set.
We can now define a complete set of Horn rules (though it’s huge, and somewhat cheating).

Each tuple \((N_0, \ldots, N_k)\) defines a sound Horn rule, which is simply the one whose side-condition is the inconsistencies required by plugging these values into the table. The set of rules generated by all tuples is a complete set of Horn rules - in fact all sound rules could be derived in a single step!

We can improve this a little - it turns out we only need consider values of \(N_i\) that are rational functions of \(q\). But in fact almost all rules of a given size reduce to a small set. The only sound rules with two or fewer premises are those defined by the tuples:

\[
(1) \quad (1,1) \quad (1,1,1) \quad (1,1,q)
\]
It turns out these four rules are exactly equivalent to the rules of $O$.

However all is not well. The $(2, 1, 1, 1)$ rule shows that (when $p, q, r$ are propositional variables, say)

$$\text{if } (e \land \neg f) \lor (\neg e \land f) \models g, \ e \models g \text{ and } f \models g \text{ then } e \lor f \not\models g$$

but is not derivable from $O$.

In fact, take any $k \geq 2$ and consider the rule generated by the tuple $(k, 1, 1, \ldots, 1)$, where there are $(k + 1)$ 1’s. It can be shown, by examining its side conditions, that this rule can deduce conditionals that cannot ever be reached by a deduction using rules with $k$ or fewer premises.

There is therefore no complete finite set of Horn rules for PCRs.

