Unform Properties
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1 Motivation

Theorem 1.1. (G. Keller, 1972) Let \( V \) be a variety of groups. Then \( V \) is uniformly amenable iff every group in \( V \) is amenable.

- What does uniformly amenable mean?
- What properties of variety is used here? Does it hold for other classes of groups?
- Are there similar results for mathematical properties other than amenability, or objects other than groups? What is really going on here?
2 Uniform properties: not-groups

2.1 Continuous functions

Everyone knows the following:

**Proposition 2.1.** Let \( f : \mathbb{R} \to \mathbb{R} \). TFAE:

1. \( f \) is uniformly continuous

2. \( f \) is (uniformly) S-continuous:

   \[ \forall x, y \in \star \mathbb{R} \ x \approx y \Rightarrow f(x) \approx f(y) \]

3. \( f \) is (pointwise) S-continuous:

   \[ \forall x \in \star \mathbb{R} \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall y \in \star \mathbb{R} \ |y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon \]
Similarly:

**Proposition 2.2.** Let $F$ be a set of real functions. TFAE:

1. $F$ is uniformly equicontinuous, that is, for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $f$ in $F$ and $x, y \in \mathbb{R}$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

2. Every $f \in F$ is pointwise $S$-continuous

**Remarks 2.1.** These results have the following form:

- For a mathematical object $O$: If $O$ has property $P$ then $O$ is uniformly $P$.

- For a set $S$ of such objects: If every $s \in S$ has property $P$ then $S$ is uniformly $P$. 
2.2 Statistical experiments

Recall that a statistical experiment is a parametrized family $\mathcal{E} = (X, \mathcal{A}, P_\theta)_{\theta \in \Theta}$ of probability measures on a measurable space.

The experiment $\mathcal{E}$ is homogeneous provided $\forall \theta, \theta' \in \Theta, P_\theta \ll P_{\theta'}$.

Let $\hat{\mathcal{E}} = (\hat{X}, \mathcal{A}_L, (P_\theta)_L)_{\theta \in \Theta}$ (Note: not the “nonstandard hull of an experiment”).

**Proposition 2.3.** If $\hat{\mathcal{E}}$ is homogeneous then so is $\mathcal{E}$.

The converse need not hold; for example, if $P_\theta$ is a normal distribution on $\mathbb{R}$ with mean $0$ and variance $\theta$ for $\theta > 0$, then $\mathcal{E}$ is homogeneous but $\hat{\mathcal{E}}$ is not: $(P_\theta)_L(\text{monad}(0)) = 0$ or $1$ depending as $\theta \approx 0$ or $\theta \not\approx 0$. 
We could simply define “uniform homogeneity” for statistical experiments by

\( \mathcal{E} \) is uniformly homogeneous provided \( \hat{\mathcal{E}} \) is homogeneous

but there is an equivalent standard definition:

**Proposition 2.4.** Let \( \mathcal{E} \) be a statistical experiment. TFAE:

1. \( \hat{\mathcal{E}} \) is homogeneous

2. \( \forall \epsilon > 0 \exists \delta > 0 \forall \theta, \theta' \in \Theta \forall E \in \mathcal{A} \ P_{\theta}(E) < \delta \Rightarrow P_{\theta'}(E) < \epsilon \)
3 Uniform properties: groups

3.1 Amenability

A group $G$ is amenable if there is a left-invariant, finitely-additive probability measure $\mu$ on $(G, \mathcal{P}(G))$ with $P(G) = 1$. (Call such a $\mu$ a mean.)

Example 3.1. $\mathbb{Z}$ is amenable under addition.

Proof. Let $H \in^{*} \mathbb{N}$ be infinite, and put $I = [-H, H]$. For $A \subseteq \mathbb{Z}$ let $P(A) = q\left(\frac{1}{2H+1} \cap^{*} A\right)$. $P$ is a finitely-additive measure on $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$, and if $0 \neq a \in \mathbb{Z}$ then the difference between $|I \cap^{*} A|$ and $|I \cap^{*}(a + A)|$ is $\leq 2a$; it follows that $P(A) - P(a + A) = 0$. \hfill \Box

Example 3.2. Let $F_{2}$ be the free group on two generators, i.e., all formal words on the letters $\{a, b, a^{-1}, b^{-1}\}$ with the only simplification rules being the obvious ones. Then $F_{2}$ is not amenable.

Example 3.3. Finite groups are amenable; abelian groups are amenable; homomorphic images and subgroups of amenable groups are amenable;...
Theorem 3.1. Let \((G,e)\) be a multiplicative group. Denote by \(L^\infty(G)\) the bounded real functions on \(G\). The following are equivalent: (a) \(G\) is amenable; (b) there is a positive linear functional \(T : L^\infty(G) \to \mathbb{R}\) which is \(G\)-invariant in the sense that for any \(g \in G\) and \(f \in L^\infty(G)\), \(T(f) = T(f \circ \phi_g)\), where \(\phi_g\) is the function \(a \mapsto ga\) from \(G\) to \(G\).

Proof. \((b \Rightarrow a)\) is trivial.

\((a \Rightarrow b)\) Via Loeb measures: Given \(\mu\), define a functional by \(T(f) = \int_G f \, d\mu\). It is easy to confirm that \(T\) works. \(\square\)

Other interesting equivalences:

Theorem 3.2. (Følner): \(G\) is amenable if and only if:

\[
\forall A \subseteq G \text{ finite } \forall r < 1 \exists E \subseteq G \text{ finite } \forall a \in A \quad \frac{|E \cap aE|}{|E|} > r
\]

(Henson has given a nice nonstandard proof of \(\Leftarrow\).)

Theorem 3.3. (Kesten): \(G\) is amenable if and only if: For any finite symmetric subset \(A\) of \(G\), \(\frac{2^{\sqrt{n}}B_{2n}}{n} \to |A|\) as \(n \to \infty\), where \(B_k\) is the number of formal words of length \(k\) from \(A\) which reduce to the identity.
A useful fact with a nice nonstandard proof:

**Theorem 3.4.** Let $G$ be a group. TFAE:

1. $G$ is amenable
2. Every subgroup of $G$ is amenable
3. Every finitely-generated subgroup of $G$ is amenable

**Proof.** $(2 \Rightarrow 3)$ is trivial. $(3 \Rightarrow 1)$ Let $S$ be a hyperfinite subset of $G$ which contains $G$, let $S'$ and let $\hat{G}$ be the internal $\ast$--group generated by $S$. By transfer $\hat{G}$ is $\ast$--amenable; let $\mu$ be a $\ast$--mean. For $A \subseteq G$ let $\nu(A) = \Diamond \mu(A)$. It is easy to verify that $\mu$ is a mean on $G$.

$(1 \Rightarrow 2)$ Standard and tedious, involving coset representations. \(\square\)
If $G$ is a group, then $\ast G$ is not only a *group, but also a group. Does amenability of one imply amenability of the other?

**Theorem 3.5.** If $\ast G$ is amenable then so is $G$. In fact, if there is an amenable subgroup $G'$ of $\ast G$ with $G \subseteq G'$ then $G$ is amenable.

**Proof.** Let $\mu$ be an mean on $G'$. Define $\nu$ on $(G, \mathcal{P}(G))$ by $\nu(A) = \mu(G' \cap \ast A)$, it is easy to verify that $\nu$ is a mean. $\square$

The converse does not hold:

**Example 3.4.** Let $G = \{\pi \in \text{Permutations}(\mathbb{N}) : \exists N \in \mathbb{N} \ \forall x > N \ \pi(x) = x\}$. Then $G$ is amenable, but $\ast G$ is not.

Amenability of $G$: every finitely-generated subgroup of $G$ is finite, so amenable, and by the above $G$ is amenable. Nonamenability of $\ast G$: find a copy $F_2$ in the set of permutations of $\{0, 1, \ldots, H-1\}$ for some infinite $H-1$. 
A group $G$ is uniformly $F\Delta$lner, or uniformly amenable (UA) if $|E|$ can be chosen to depend only on $|A|$ and $r$, that is, if there is a function $F : \mathbb{N} \times (0,1) \to \mathbb{N}$ such that

$$\forall n \in \mathbb{N} \ \forall A \subseteq G \text{ s.t. } |A| < n \ \forall r < 1$$

$$\exists E \subseteq G \text{ s.t. } |E| < F(n,r) \ & \ \forall a \in A \frac{|E \cap aE|}{|E|} > r$$

A class $\mathcal{D}$ of groups is uniformly amenable if there is a single function $F : \mathbb{N} \times (0,1) \to \mathbb{N}$ that witnesses UA for all the groups in $\mathcal{D}$.

**Theorem 3.6.** Let $G$ be a group; then $^*G$ is amenable iff $G$ is uniformly amenable. More generally, let $\mathcal{G}$ be a set of groups; then every $G \in ^*\mathcal{G}$ is amenable iff $\mathcal{G}$ is uniformly amenable.

Proof of ($\Leftarrow$): Fix $n \in \mathbb{N}, r < 1$. We need to define $F(n,r)$. Let $m \in ^*\mathbb{N} \setminus \mathbb{N}$. By amenability of $^*\mathcal{G}$ and the $F\Delta$lner condition, $m$ witnesses

$$\exists m \in ^*\mathbb{N} \ \forall G \in ^*\mathcal{G} \ \forall A \in ^*\mathcal{P}(G) |A| < n \Rightarrow \exists E \in ^*\mathcal{P}(G), \ |E| \text{ finite } \& \ \forall a \in A \frac{|E \cap aE|}{|E|} > r$$

as above. By transfer, there is a standard finite $m$ that works for this $n$ and $r$; put $F(n,r) := m$. 
3.2 Generation of groups

$H \subseteq G$ generates a group $G$ if $G$ is the smallest subgroup of $G$ which contains $H$.

If $e \in H = H^{-1}$, then $H$ generates $G$ provided $G = \bigcup_n H^n$

**Proposition 3.1.** for a group $G$ and subset $H$ the following are equivalent:

1. $H$ generates $G$

2. $G = \bigcup_{n<N}(H \cup H^{-1})^n$ for some finite $N$

More generally, if $\mathcal{D}$ is a set of pairs of groups and subsets, then every group in $^*\mathcal{D}$ is generated by the corresponding subset if and only if for some $N$ and every $(G,H) \in \mathcal{D}$, $G = \bigcup_{n<N}(H \cup H^{-1})^n$. 
4 Quasivarieties

Call a class $\mathcal{C}$ of mathematical objects a quasivariety provided whenever $\mathcal{D} \subseteq \mathcal{C}$ is a set then $\star \mathcal{D} \subseteq \mathcal{C}$.

Example 4.1. A variety of groups (i.e., class of all groups satisfying a finite set of identity relations) is a quasivariety. More generally, the class of all groups satisfying an arbitrary set of identity relations is a quasivariety.

Example 4.2. For any fixed $\epsilon > 0$, pairs $(G, H)$, where $G$ is a group, $H$ is a subset that generates it, and there exists a left-invariant finitely-additive probability measure $\mu$ on $(G, \mathcal{A})$ where $H \in \mathcal{A}$ and $\mu(H) > \epsilon$. (Denote this set of pairs by $\mathcal{D}_\epsilon$)
Theorem 4.1. Let \( \mathcal{C} \) be a class of mathematical objects, \( P \) be a property of form \( P(\mathcal{O}) \Leftrightarrow \forall i \in I \exists j \in J \phi_{i,j}(\mathcal{O}) \) where \( J \) is a directed set, and suppose that \( \mathcal{V} \subseteq \mathcal{C} \) is a quasivariety such that every \( \mathcal{O} \) in \( \mathcal{V} \) satisfies \( P \). Then \( \mathcal{V} \) is “uniformly \( P \)” in the sense that

\[
\forall i \in I \ \exists j \in J \ \forall \mathcal{O} \in \mathcal{V} \ \phi_{i,j}(\mathcal{O})
\]

Corollary 4.1. If every group in a variety \( \mathcal{V} \) is amenable, then \( \mathcal{V} \) is uniformly amenable.

Corollary 4.2. Fix \( \epsilon > 0 \). Then for some \( N \) and every \( (G, H) \in D_{\epsilon} \), \( G = \bigcup_{n<N}(H \cup H^{-1})^n \)

(Proof requires an argument that for \( (G, H) \in D_{\epsilon} \), \( H \) generates \( G \). One such argument is due to vDD.)
Proof of Theorem:

First, note that the usual proof shows that a set $D$ is uniformly $P$ iff every element of $D$ is $P$.

Now, if the theorem fails,

$$\exists i \in I \ \forall j \in J \ \exists O_j \text{ such that } \neg \phi_{i,j}(O_j)$$

Fix $i$, let $D = \{O_j\}_{j \in J}$.

Evidently $D$ is not uniformly $P$, so not all elements of $D$ are $P$.

This contradicts the assumption that $\mathcal{V}$ is a quasivariety with every element satisfying $P$. 