‘CENTRAL LIMIT THEOREM FOR ASSOCIATED CLASS FUNCTIONS ON THE SYMMETRIC GROUP’

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CENTRAL LIMIT THEOREM FOR ASSOCIATED CLASS FUNCTIONS ON THE SYMMETRIC GROUP

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Abstract. Hambly, Keevash, O’Connell and Stark have proven a central limit theorem for the characteristic polynomial of a permutation matrix with respect to the uniform measure on the symmetric group. We generalize this result in several ways. We prove here a central limit theorem for associated class functions on symmetric group with respect to the Ewens measure and compute the covariance of the real and the imaginary part in the limit. We also estimate the rate of convergence with the Wasserstein distance.

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1. Introduction

The study of random matrices has gained importance in many areas of mathematics and physics, for example in nuclear physics, infinite dimensional integrable systems and large-n representation theory. Random matrix theory (RMT) was in the recent years also of big interest in number theory since the study of the spectrum of the characteristic polynomial of a random matrix in a compact Lie group was central in obtaining conjectures. A good example to illustrate this is the paper of Keating and Snaith [10]. They conjectured that the Riemann zeta function on the critical line could be modeled by the characteristic polynomial of a random unitary matrix considered on the unit circle. One of the results in [10] is

**Theorem 1.1.** Let \( x \) be a fixed complex number with \( |x| = 1 \) and \( g_n \) be a unitary matrix chosen at random with respect to the Haar measure. Then

\[
\frac{\log(\det(I_n - xg_n))}{\sqrt{\frac{1}{2} \log(n)}} \xrightarrow{d} N_1 + iN_2 \text{ for } n \to \infty
\]

and \( N_1, N_2 \) independent, normal distributed random variables.

Constin and Lebowitz have proven 10 years earlier in [6] a weaker version of this theorem. They showed that

\[
\text{Im} \left( \frac{\log(\det(I_n - xg_n))}{\sqrt{\frac{1}{2} \log(n)}} \right) \xrightarrow{d} N.
\]

They conjectured that the same is true for the real part and that the imaginary part and the real part are independent in the limit, but haven’t been able to prove this.

The situation for the characteristic polynomial of permutation matrices is similar. A permutation matrix is a unitary matrix of the form \( (\delta_{i,\sigma(j)})_{1 \leq i,j \leq n} \) with \( \sigma \in S_n \) and \( S_n \) the symmetric group. It is easy to see that the permutation matrices form a group isomorphic to \( S_n \). We call for simplicity both groups \( S_n \) and use this identification without mentioning it explicitly.

The characteristic polynomial \( Z_n(x) \) of a permutation matrix is defined as

\[
Z_n(x) = Z_n(x)(\sigma) := \det(I - x\sigma) \text{ with } x \in \mathbb{C}, \sigma \in S_n.
\]

Hambly, Keevash, O’Connell and Stark have proven in [9]

**Theorem 1.2 (B.M.Hambly, P.Keevash, N.O’Connell and D.Stark).** Let \( g \in S_n \) be chosen uniformly at random and \( x \) be a fixed complex number with \( |x| = 1 \), not a root of unity and of finite type. Then

\[
\text{Re} \left( \frac{\log(Z_n(x))}{\sqrt{\frac{1}{12} \log(n)}} \right) \xrightarrow{d} N_a, \quad \text{Im} \left( \frac{\log(Z_n(x))}{\sqrt{\frac{1}{12} \log(n)}} \right) \xrightarrow{d} N_b
\]

with \( N_a, N_b \) standard normal distributed random variables.

As for unitary matrices, it is natural to ask if \( N_a \) and \( N_b \) are independent. This question is not considered in [9]. We will prove in corollary 2.8.1 that \( N_a \) and \( N_b \) are indeed independent. This is a conclusion of the main theorem of this paper, theorem 2.8. Theorem 2.8 is an extension of theorem 1.2 with three important differences. These differences are
We compute the correlation of the real and the imaginary part in the limit.

We endow $S_n$ with the Ewens measure (see definition 2.4), which is a generalization of the uniform measure.

We consider more general class functions on $S_n$, the so called associated class functions $W^n(f)$ (see definition 2.6).

We have introduced associated class functions $W^n(f)$ in [7] as generalization of $Z_n(x)$ and studied there the asymptotic behavior of their moments with respect to the uniform measure on $S_n$.

The main idea there is to write down the generating function with a combinatorial argument and to use function theory to extract the asymptotic behavior.

The structure of this paper is as follows: we introduce in section 2 associated class functions $W^n(f)$ and state the main theorem of this paper. We do in section 3 some preparations and state in section 4 an auxiliary central limit theorem. We then prove in section 5 the main theorem 2.8. In section 6 we then estimate the convergence rate of the probability measures in the main theorem with the Wasserstein distance.

2. Definition and central limit theorem

We introduce in section 2.1 the Ewens measure and some well known functions on $S_n$. In section 2.2 we give an alternative expression for $Z_n(x)$ and use this expression to introduce the associated class functions $W^n(f)(x)$. We then state in section 2.3 the main theorem of this paper.

2.1. The symmetric group $S_n$. All functions appearing in this paper are class functions ($u(high^{-1}) = u(g))$ on $S_n$. It is therefore natural to take a look at the conjugation classes of $S_n$. We parameterize them with partitions.

Definition 2.1. A partition $\lambda$ is a sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ eventually trailing to 0, which we usually omit. We use the notation $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$.

The length of $\lambda$ is the largest $l$ such that $\lambda_l \neq 0$. We define the size $|\lambda| := \sum \lambda_i$ and we set for $n \in \mathbb{N}$

$$\lambda \vdash n := \{\lambda \text{ partition}; |\lambda| = n\}.$$ 

Let $\sigma \in S_n$ be arbitrary and write $\sigma = \sigma_1 \sigma_2 \cdots \sigma_l \in S_n$ with $\sigma_i$ disjoint cycles of length $\lambda_i$. Since disjoint cycles commute, we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. We call the partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ the cycle type of $\sigma$. It remains to show that two elements of $S_n$ are conjugated if and only if they have the same cycle type. Since this is well known, we omit the proof here and refer to [4] chapter 39 for a proof.

We now introduce the number of cycles of a given length. We set

Definition 2.2. Let $\sigma \in S_n$ be given with cycle-type $\lambda = (\lambda_1, \cdots, \lambda_l)$. We define

$$C_m := C_m^n := C_m^n(\sigma) := \# \{i | 1 \leq i \leq l \text{ and } \lambda_i = m\}.$$ 

The functions $C_m^n$ depends only on the cycle-type of $\sigma$ and are therefore class functions on $S_n$.

It is clear that the cycle type of $\sigma \in S_n$ is uniquely determined by the values of $C_1^n, \cdots, C_l^n$. We therefore can work with the functions $C_m^n$ or with partitions. We prefer here to use $C_m^n$.

The most natural measure on $S_n$ is the uniform measure (i.e $P[A] = \frac{|A|}{n!}$). The probability distribution and the expectation of $C_m^n$ with respect to the uniform measure is given by
Lemma 2.3. Let $c_1, c_2, \ldots, c_n \in \mathbb{N}$ be given and $S_n$ endowed with the uniform measure. Then
\begin{equation}
P[(C_1 = c_1, \ldots, C_n = c_n)] = \prod_{m=1}^{n} \frac{1}{c_m!} \left( \frac{1}{m} \right)^{c_m} 1 \left( \sum_{m=1}^{n} mc_m = n \right).
\end{equation}

The expectation of $C_m^{(n)}$ is given by
\begin{equation}
E \left[ C_m^{(n)} \right] = \frac{1}{m} 1(m \leq n).
\end{equation}

Proof. See [3, page 3].

There exists of course other important measures on $S_n$ than the uniform measure. One of this measure is the so called Ewens measure, appearing in population genetics. The Ewens measure is a generalization of the uniform measure and has an additional weight depending on the total number of cycles. Many results about the Ewens measure can be found in the book [3, chapter 4]. We state here only the definitions and results we need and refer to [3, chapter 4] for the proofs.

We begin with

Definition 2.4. Let $\theta > 0$ be given. The probability distribution of $(C_1, C_2, \ldots, C_n)$ with respect to the Ewens measure on $S_n$ with parameter $\theta$ is given by
\begin{equation}
P_{\theta}[(C_1 = c_1, \ldots, C_n = c_n)] = \frac{n!}{\theta(\theta + 1) \cdots (\theta + n - 1)} \prod_{m=1}^{n} \frac{1}{c_m!} \left( \frac{\theta}{m} \right)^{c_m} 1 \left( \sum_{m=1}^{n} mc_m = n \right)
\end{equation}

We write $E_{\theta} [\cdot]$ and $C_{m,\theta} = C_m^{(n)}$ to emphasized the dependence on $\theta$ if necessary. If $\theta$ is clear from the context, we just write $P[\cdot], E[\cdot]$ and $C_m^{(n)}$.

The expectation of $C_m^{(n)}$ can also be computed. We have

Lemma 2.5.
\begin{equation}
E \left[ C_m^{(n)} \right] = \frac{\theta}{m} \frac{\theta + n - m - 1}{\theta + n - 1} 1(m \leq n).
\end{equation}

We use here the definition $\binom{\alpha}{m} := \frac{\prod_{k=0}^{m-1}(\alpha-k)}{m!}$ for $\alpha \in \mathbb{C}$.

Proof. See [3, chapter 4, (4.7)]

2.2. Definition of $W^n(f)$. We are now ready to introduce associated class functions. We first give a more explicit expression for $Z_n(x)$ using the cycle type. Let $\sigma \in S_n$ be given with cycle type $\lambda = (\lambda_1, \lambda_2, \cdots)$. Then
\begin{equation}
Z_n(x)(g) = \prod_{i=1}^{l(\lambda)} (1 - x^{\lambda_i}).
\end{equation}

The proof of this equation is straight forward. One first has to take a look a the case when $\sigma$ is a cycle and then take a look at the general case. More details can be found in [13].

We prefer to work here with $C_m^{(n)}$ and therefore reformulate (2.6). We have
\begin{equation}
Z_n(x) = \prod_{m=1}^{n} (1 - x^m)^{C_m^{(n)}}.
\end{equation}
This follows immediately from (2.6) since we have by definition \( C^{(n)}_{m}(g) = \# \{ \lambda_i; \lambda_i = m \} \).

We now define associated class functions.

**Definition 2.6.** Let \( f : S^1 \to \mathbb{C} \) be real analytic. We then define

\[
W^n(f) = W^n(f)(x) := \prod_{m=1}^{n} f(x^m)^{C^{(n)}_{m}}.
\]

We call \( W^n(f) \) the associated class function of \( f \).

We also set

\[
w^n(f)(x) = \log(W^n(f)(x)) := \sum_{m=1}^{n} C^{(n)}_{m} \log(f(x^m))
\]

with \( \log \) the principal branch of logarithm and \( \log(-y) := \log(y) + i\pi \) for \( y \in \mathbb{R}_{>0} \) and \( \log(0) = \infty \).

It is clear that \( Z_n(x) = W^n(1 - x) \) and that \( W^n(f) \) and \( w^n(f) \) are class functions on \( S_n \).

**2.3. Main theorem.** Before we can formulate the main result of this paper, we have to define

**Definition 2.7.** Let \( f : S^1 \to \mathbb{C} \) be a real analytic function, \( x \in S^1 \) be arbitrary but fixed. We set

\[
m(f)(x) = \begin{cases} 
\int_0^1 \log(f(e^{2\pi i s})) \, ds & \text{if } x \text{ is not a root of unity and the integral exists,} \\
\frac{1}{p} \sum_{m=1}^{p} \log(f(x^m)) & \text{if } x \text{ is a root of unity of order } p \text{ and the sum exists,} \\
\infty & \text{otherwise.}
\end{cases}
\]

We know that a real analytic function \( f \neq 0 \) has only isolated zeros and \( \log(f(e^{2\pi i s})) \sim C \log(s - s_0) \) for \( s \to s_0 \) and \( s_0 \) a zero of \( f(e^{2\pi i s}) \). Therefore the integral in the definition exists for all \( f \neq 0 \) and \( m(f)(x) = \infty \) if and only if \( f \equiv 0 \) or \( x \) is a root of unity and \( f(x^m) = 0 \) for some \( m \in \mathbb{N} \). One can rewrite the integral in the definition of \( m(f) \) as \( \int_{S^1} \log(f(\varphi)) \, d\varphi \) where \( d\varphi \) is the uniform measure on \( S^1 \). We have not used this because this can be confounded with the complex integral \( \int_{S^1} \log(f(z)) \, dz = 2\pi i \int_0^1 \log(f(e^{2\pi i s})) e^{2\pi i s} \, ds \).

We now come to the main theorem.

**Theorem 2.8.** Let \( f : S^1 \to \mathbb{C} \) be real analytic and \( x \in S^1 \). Let \( w(n)(f) \) be distributed according to the Ewens measure with parameter \( \theta > 0 \). We then have

- If \( x \) is not a root of unity, of finite type and all zeros of \( f \) are roots of the unity then

\[
w^n(f)(x) = \frac{\theta}{\sqrt{\log(n)}} \log(n) m(f)(x) + i N_b
\]

with \( N_a \) and \( N_b \) normal distributed random variables. The covariance of \( N_a \) and \( N_b \) is given by

\[
\frac{1}{2} \text{Im} \left( \int_0^1 \log^2(f(e^{2\pi i s})) \, ds \right).
\]

\( N_a \) and \( N_b \) are independent if and only if the covariance is equal to 0.
If \( x \) is a root of unity of order \( p \) and \( f(x^m) \neq 0 \) for all \( 1 \leq m \leq p \) then

\[
\frac{w^n(f)(x)}{\sqrt{\log(n)}} - \theta \sqrt{\log(n)m(f)} \xrightarrow{d} N_1 + i N_2
\]

with \( N_a \) and \( N_b \) normal distributed random variables. The covariance of \( N_a \) and \( N_b \) is given by

\[
\frac{1}{2} \Im \left( \frac{1}{p} \sum_{m=1}^{p} \log^2 \left( f(x^m) \right) \right).
\]

\( N_a \) and \( N_b \) are independent if and only if the covariance is equal to 0.

Intuitively, the condition \( x \) of finite type ensures that the sequence \( (x^m)_{m=1}^{n} \) does not approach too fast to the zeros of \( f \) for \( n \to \infty \). This condition is essential in our proof. We postpone the precise definition of finite type to section 3.4 since we can illustrate there the necessity of this condition for our proof.

As promised in the beginning, we now can prove

**Corollary 2.8.1.** The random variables \( N_a \) and \( N_b \) in theorem 1.2 are independent.

**Proof.** We know that \( Z_n(x) = W^n(1 - x) \). We therefore can apply theorem 2.8. We thus can prove independence if we can show that (2.12) is equal to 0 for \( f(x) = 1 - x \). A simple computation shows

\[
\log(1 - e^{2\pi is}) = \log |1 - e^{2\pi is}| - i\pi s \text{ for } s \in [-\frac{1}{2}, \frac{1}{2}].
\]

We get

\[
\frac{1}{2} \Im \left( \int_{0}^{1} \log^2 (1 - e^{2\pi is}) \, ds \right) = \int_{-1/2}^{1/2} \Re(\log(1 - e^{2\pi is})) \Im(\log(1 - e^{2\pi is})) \, ds
\]

\[
= \int_{-1/2}^{1/2} -s \pi \log |1 - e^{2\pi is}| \, ds = 0
\]

This integral is equal to 0 since the integrand is odd. \( \square \)

3. **Preparations**

Before we can prove theorem 2.8, we have to do some preparations.

3.1. **Asymptotic behavior of \( C_{m}^{(n)} \) and the Feller-coupling.** We take here a short look at the functions \( C_{m}^{(n)} \) with parameter \( \theta > 0 \) fixed. As mentioned above, these functions are well studied objects and many results for them can be found in [3]. In particular, the asymptotic behavior of \( C_{m}^{(n)} \) is also well known. We have

**Lemma 3.1.** The random variables \( C_{m}^{(n)} \) converge for each \( m \in \mathbb{N} \) in distribution to a Poisson distributed random variable \( Y_m = Y_{m,\theta} \) with \( \mathbb{E}[Y_m] = \frac{\theta}{m} \). In fact, we have for all \( b \in \mathbb{N} \)

\[
(C_{1}^{(n)}, C_{2}^{(n)}, \ldots, C_{b}^{(n)}) \xrightarrow{d} (Y_1, Y_2, \ldots, Y_b) \quad (n \to \infty),
\]

with all \( Y_m \) independent.
One of the problems in the definition of convergence in distribution of a sequence \((X_n)_{n \in \mathbb{N}}\) is that all \(X_n\) can be defined on a different probability space. Therefore it is very difficult to compare \(X_n\) with \(X_{n+1}\) directly. This is the case for \(C_m^{(n)}\) and \(C_m^{(n+1)}\).

Fortunately, the Feller coupling constructs for each \(\theta > 0\) a probability space and new random variables \(C_m^{(n)}\) and \(Y_m\) on this space, which have the same distributions as the \(C_m^{(n)}\) and \(Y_m\) above and can easily be compared. Many details on the Feller coupling can be found in [3, chapter 4]. We state here only the things we need.

The construction works as follows: Let \(\xi := (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \cdots)\) be a sequence of independent Bernoulli-random variables with \(E[\xi_m] = \frac{\theta}{(\theta - 1) + m}\). An \(m\)-spacings is a sequence of \(m - 1\) consecutive zeroes in \(\xi\) or its truncations:

\[
\begin{array}{cccccc}
1 & 0 & \cdots & 0 & 1 \\
\end{array}
\]

\(m - 1\) times

**Definition 3.2.** Let \(C_m^{(n)}(\xi)\) be the number of \(m\)-spacings in \(1\xi_2\cdots\xi_{n+1}\). We define \(Y_m(\xi)\) to be the number of \(m\)-spacings in the whole sequence \(\xi\).

**Theorem 3.3.** We have

- The above-constructed \(C_m^{(n)}(\xi)\) have the same distribution as the \(C_m^{(n)}(\lambda)\) in definition 2.2.
- \(Y_m(\xi)\) is a.s. finite and Poisson distributed with \(E[Y_m(\xi)] = \frac{\theta}{m}\).
- All \(Y_m(\xi)\) are independent.
- We have

\[
E_\theta \left[ |C_m^{(n)}(\xi) - Y_m(\xi)| \right] \leq \left\{ \begin{array}{ll}
\frac{\theta(\theta+1)}{\theta(\theta+1)} & \text{if } \theta \geq 1 \\
\frac{\theta}{\theta + 1} & \text{if } 0 < \theta < 1.
\end{array} \right.
\]

- For any fixed \(b \in \mathbb{N}\),

\[
P \left[ (C_1^{(n)}(\xi), \cdots, C_b^{(n)}(\xi)) \neq (Y_1(\xi), \cdots, Y_b(\xi)) \right] \to 0 \text{ (}n \to \infty\text{)}.
\]

**Proof.** See [3, chapter 4] and [2, theorem 2].

We write \(C_m^{(n)}\) and \(Y_m\) for both sets of random variables and do not distinguish them anymore.

### 3.2. Elementary analysis

We give here some simple results from analysis that we need. We state them without further comments.

**Lemma 3.4 (Abel’s partial summation).** Let \(a_1, \cdots, a_n, b_1, \cdots, b_n\) be complex numbers. Then

\[
\sum_{m=1}^{n} a_m b_m = A_n b_n + \sum_{m=1}^{n-1} A_m (b_{m+1} - b_m)
\]

with \(A_m := \sum_{k=1}^{m} a_k\) for \(m \geq 1\) and \(A_0 := 0\).

**Proof.** The idea of the proof is to write \(a_m = A_{m+1} - A_m\) in the first sum and then split the sum. More details can be found in [8].

We can use this to prove
Lemma 3.5. Let \((a_m)_{m=1}^n\) be a finite sequence and \(A(s) := \sum_{m \leq s} a_m\). We then have
\[
\sum_{m=1}^n \frac{a_m}{m} = \left( \frac{1}{n} \sum_{m=1}^n a_m \right) + \int_1^n A(s) \frac{1}{s^2} \, ds
\] (3.4)

Proof. Use lemma 3.4 with \(b_m = \frac{1}{m}\). The rest is a straightforward verification. \(\square\)

Lemma 3.6. Let \((a_m)_{m=1}^\infty\) be a complex sequence. If we have \(\frac{1}{n} \sum_{m=1}^n a_m = E + O(n^{-\delta})\) for some \(\delta > 0\), then
\[
\sum_{m=1}^n \frac{a_m}{m} = E \log(n) + \text{Const.} + O(n^{-\delta})
\] (3.5)
\[
\sum_{n/2 < m \leq n} \frac{a_m}{m} = E \log(2) + O(n^{-\delta})
\] (3.6)

Proof. We use lemma 3.5 and get
\[
\sum_{m=1}^n \frac{a_m}{m} = \left( \frac{1}{n} \sum_{m=1}^n a_m \right) + \int_1^n \left( \sum_{m \leq s} a_m \right) \frac{1}{s^2} \, ds
\]
\[
= \left( E + O(n^{-\delta}) \right) + \int_1^n \left( sE + O(s^{1-\delta}) \right) \frac{1}{s^2} \, ds
\]
\[
= E \log(n) + E + O(n^{-\delta}) + \int_{n/2}^n O(s^{-1-\delta}) \, ds
\] (3.7)

This shows the first identity. The second follows immediately with
\[
\sum_{n/2 < m \leq n} \frac{a_m}{m} = \left( \sum_{m=1}^n (...) - \sum_{1 \leq m \leq n/2} (...) \right)
\]
\[
= E \left( \log(n) - \log([n/2]) \right) + O(n^{-\delta}) + \int_{n/2}^n O(s^{-1-\delta}) \, ds
\]
\[
= E \log(2) + O(n^{-\delta})
\] (3.8)

Finally we need

Lemma 3.7 (Hölder inequality). Let \((a_m)_{m=1}^n\) and \((b_m)_{m=1}^n\) be finite sequences and \(p, q \geq 1\) such that \(\frac{1}{p} + \frac{1}{q} = 1\). Then
\[
\sum_{m=1}^n |a_m b_m| \leq \left( \sum_{m=1}^n |a_m|^p \right)^{1/p} \left( \sum_{m=1}^n |b_m|^q \right)^{1/q}
\] (3.9)

3.3. Uniformly distributed sequences. We need some facts about uniformly distributed sequences. We follow the book [11]. We omit here most of the proofs since they are not difficult and can be found in the book [11] or my Ph.D thesis [14, Chapter 2].
3.3.1. Definition and properties. We begin with the definition of uniformly distributed sequences.

**Definition 3.8.** Let \( t = (t_m)_{m=1}^\infty \) be a sequence of real numbers in the compact interval \([a, b]\) (with \(a < b\)). We put for \(a \leq \alpha \leq \beta \leq b\)

\[
A(\alpha, \beta, n) = A(\alpha, \beta, n, t) := \# \{1 \leq m \leq n; t_m \in [\alpha, \beta]\}
\]

(3.10)

The sequence \( t = (t_m)_{m \in \mathbb{N}} \) is called uniformly distributed in \([a, b]\) if we have

\[
\lim_{n \to \infty} \left| \frac{A(\alpha, \beta, n)}{n} - \frac{(\beta - \alpha)}{b - a} \right| = 0
\]

(3.11)

for each \(\alpha, \beta\) with \(a \leq \alpha \leq \beta \leq b\).

It is usual to work on the interval \([0, 1]\). We have introduced here this more general version since we have to restrict our sequences to subintervals (see definition 3.17).

The following theorem shows that the name uniformly distributed is well chosen.

**Theorem 3.9.** Let \( h : [a, b] \to \mathbb{C} \) be a Riemann-integrable function and \( t = (t_m)_{m \in \mathbb{N}} \) be a uniformly distributed sequence in \([a, b]\). Then

\[
\frac{1}{n} \sum_{m=1}^{n} h(t_m) \to \frac{1}{b - a} \int_{a}^{b} h(s) \, ds.
\]

(3.12)

**Proof.** We first look at function of the form \( I_{\alpha, \beta}(s) := \begin{cases} 1 & \text{if } \alpha \leq s \leq \beta, \\ 0 & \text{otherwise.} \end{cases} \) with \( a \leq \alpha < \beta \leq b \).

The theorem is true for such functions since \( \frac{1}{b - a} \int_{a}^{b} I_{\alpha, \beta}(s) \, ds = \frac{(\beta - \alpha)}{b - a} \). The theorem now follows by an approximation argument. \(\square\)

We are primary interested in sequences of the form \((\{mt\})_{m=1}^\infty\) with

\[
\{t\} := t - \lfloor t \rfloor \quad \text{and} \quad \lceil t \rceil := \max \{n \in \mathbb{Z}; n \leq t\}.
\]

(3.13)

The next lemma shows that the sequence \((\{mt\})_{m=1}^\infty\) is for almost all \(t \in \mathbb{R}\) uniformly distributed.

**Lemma 3.10.** Let \( t \in \mathbb{R} \) be a irrational number. The sequence \((\{mt\})_{m \in \mathbb{N}}\) is uniformly distributed in \([0, 1]\).

We also introduce the definition of discrepancy.

**Definition 3.11.** Let \((t_m)_{m=1}^\infty\) be a sequence of real numbers in the compact interval \([a, b]\). We then define

\[
D_n = D_n([a, b], t) := \sup_{a \leq \alpha \leq \beta \leq b} \left| \frac{A(\alpha, \beta, n)}{n} - \frac{(\beta - \alpha)}{b - a} \right|,
\]

(3.14)

\[
D^*_n = D^*_n([a, b], t) := \sup_{a \leq \beta \leq b} \left| \frac{A(\alpha, \beta, n)}{n} - \frac{(\beta - a)}{b - a} \right|.
\]

(3.15)

We call \(D_n\) the discrepancy and \(D^*_n\) the \(*\)-discrepancy of the sequence \(t\).

It is easy to see that \(D^*_n \leq D_n \leq 2D^*_n\) and therefore \(D_n\) and \(D^*_n\) are equivalent. We prefer to work with \(D^*_n\) since we have a more explicit expression for it.
Lemma 3.12. Let $n$ be fixed, $[a, b] = [0, 1]$ and $t = (t_m)_{m \in \mathbb{N}}$ be a sequence in the interval $[0, 1]$. We define $y_1 \leq \cdots \leq y_n$ be the ascending ordered sequence $(t_m)_{m=1}^n$. We then have

\begin{equation}
D_n^*[0, 1], t) = \max_{1 \leq m \leq n} \max \left( |y_m - \frac{m}{n}|, |y_m - \frac{m-1}{n}| \right)
\end{equation}

The next lemma shows that the discrepancy is compatible with simple coordinate changes.

Lemma 3.13. Let $t = (t_m)_{m \in \mathbb{N}}$ be a sequence in the interval $[0, 1]$ and $-\infty < a < b < \infty$. We define $y_m := (b-a)t_m + a$ and set $y = (y_m)_{m \in \mathbb{N}}$. Then

\begin{align*}
(3.16) & \quad D_n([0, 1], t) = D_n([a, b], y) \\
(3.17) & \quad D_n^*([0, 1], t) = D_n^*([a, b], y)
\end{align*}

Proof. See [14, Lemma 2.3.6] \hfill \square

An important thing is that theorem 3.9, the discrepancy and uniformly distributed sequences are closely related. In fact we have

Lemma 3.14. Let $t = (t_m)_{m=1}^\infty$ be a sequence in the compact interval $[a, b]$. Then it is equivalent

1. $t$ is uniformly distributed in $[a, b]$
2. $\lim_{n \to \infty} D_n([a, b], t) = 0$
3. Let $h : [a, b] \to \mathbb{C}$ be an arbitrary Riemann-integrable function. Then

\[
\frac{1}{n} \sum_{m=1}^n h(t_m) \to \frac{1}{b-a} \int_a^b h(s) \, ds \quad \text{for } n \to \infty
\]

We have introduced the discrepancy since it allows us to estimate the rate of convergence in theorem 3.9. We have

Theorem 3.15 (Koksma’s inequality). Let $h : [a, b] \to \mathbb{C}$ be a real analytic function with $V(h) = \int_a^b \frac{d}{ds} h(s) \, ds$. Let $t = (t_m)_{m \in \mathbb{N}}$ be a arbitrary sequence in $[a, b]$. Then

\begin{equation}
\left| \frac{1}{n} \sum_{m=1}^n h(t_m) - \frac{1}{b-a} \int_a^b h(s) \, ds \right| \leq V(h)D_n^*([a, b], t).
\end{equation}

Proof. See [11]. \hfill \square

We work here with functions of the form $h(s) = \log(r(s))$ with $r(s)$ a real analytic function. A simple calculation shows that $\log(r(s)) \sim C_1 \log(s - s_0)$ and $\frac{d}{ds} \log(r(s)) \sim C_2 \frac{1}{(s - s_0)}$ for $s \to s_0$ and $s_0$ a zero of $r(s)$. We therefore cannot use Koksma’s inequality in this situation. We instead use

Theorem 3.16. Let $h : [a, b] \to \mathbb{C}$ be a real analytic function such that

\begin{equation}
\int_{a+\delta}^{b-\delta} |h(s)| \, ds < \infty \quad \text{and} \quad \int_{a+\delta}^{b-\delta} \left| \frac{d}{ds} h(s) \right| \, ds < \infty \quad \text{for all } 0 < \delta < \frac{b-a}{2}
\end{equation}

and $t = (t_m)_{m \in \mathbb{N}}$ be a sequence in $[a, b]$. Let $n \in \mathbb{N}$ be arbitrary and $\delta > 0$ such that

\[
a + \delta < \min_{1 \leq m \leq n} t_m < \max_{1 \leq m \leq n} t_m < b - \delta.
\]
We then have
\[
\left| \frac{1}{n} \sum_{m=1}^{n} h(t_m) - \frac{1}{b-a} \int_{a+\delta}^{b-\delta} h(s) \, ds \right| \leq \delta |h(a+\delta)| + \delta |h(b-\delta)|
\]
(3.21)
\[+ D_n^*[\alpha, b, t] \int_{a+\delta}^{b-\delta} \frac{d}{ds} h(s) \, ds \]

Proof. This theorem was already proven in [9] and we therefore give only a short overview. W.l.o.g. let \([a, b] = [0, 1]\). We define \(y_1 \leq \cdots \leq y_n\) be the ascending ordered sequence \((t_m)_{m=1}^{n}\). We then know from lemma 3.12 that
\[
D_n^*([0, 1], t) = \max_{1 \leq m \leq n} \max \left( \left| y_m - \frac{m}{n} \right|, \left| y_m - \frac{m-1}{n} \right| \right)
\]
We put \(y_0 := \delta, y_{n+1} := 1 - \delta\) and look at
(3.22)
\[
\sum_{m=0}^{n} \int_{y_m}^{y_{m+1}} (s - \frac{m}{n}) \frac{d}{ds} h(s) \, ds.
\]
The theorem now follows by partial integration. \(\square\)

Most of the functions we look here have the form \(\log(f(e^{2\pi is}))\). The most problematic points are of course the zeros of \(f(e^{2\pi is})\). Theorem 3.16 allows us to handle the case when the only zeros are \(s = 0\) and \(s = 1\). The idea to solve more general cases is to split the interval into several pieces such that the zeros of \(f(e^{2\pi is})\) are the boundary points and then apply theorem 3.16 separately to each piece. To do this, we have to define

Definition 3.17. Let \(t = (t_m)_{m \in \mathbb{N}}\) be a sequence in the interval \([a, b]\) and \([c, d] \subset [a, b]\). We put
\[
y_1 := t_{m_1} \quad \text{with} \quad m_1 := \min \{ m \in \mathbb{N}; t_m \in [c, d] \}
\]
\[
y_2 := t_{m_2} \quad \text{with} \quad m_2 := \min \{ m > m_1; t_m \in [c, d] \}
\]
\[
y_3 := t_{m_3} \quad \text{with} \quad m_3 := \min \{ m > m_2; t_m \in [c, d] \}
\]
\[
\vdots
\]
\[
s := \sup \{ k \in \mathbb{N}; y_k \text{ exists} \}
\]
and define \(y := (y_m)_{m=1}^{s}\). We call \(y\) the restricted sequence (to \([c, d]\)) and write \(t \cap [c, d]\) for it.

It is possible that \(y\) is finite or empty, but we will always have that \(y\) is infinite. The next lemma shows that the discrepancy and uniformly distributed sequences behaves well under restriction.

Lemma 3.18. Let \(t = (t_m)_{m \in \mathbb{N}}\) be a sequence in \([a, b]\) and \([c, d] \subset [a, b]\) with \(c < d\).

1. If \(t\) is uniformly distributed in \([a, b]\) then \(y = t \cap [c, d]\) is uniformly distributed in \([c, d]\).
2. If \(D_n([a, b], t) = O(n^{-\alpha})\) for some \(\alpha > 0\) then we have also \(D_n([c, d], y) = O(n^{-\alpha})\).

Proof. See [14] Lemma 2.3.11. \(\square\)
3.4. Diophantine approximation. As mentioned before, the sequence \( \{\{mt\}\}_{m \in \mathbb{N}} \) is the most important sequence we look at. Since our target is to use theorem 3.10, we have to choose a \( \delta = \delta(n) \) with \( \delta \leq \{mt\} \leq 1 - \delta \) for \( 1 \leq m \leq n \) and
\[
D_n^*([a, b], t) \int_{a}^{b} \left| \frac{d}{ds} h(s) \right| \, ds \to 0 \quad (n \to \infty)
\]
To reach this, we use here some classical results of diophantine approximation. We put

**Definition 3.19.** Let \( t \in \mathbb{R} \) be arbitrary. We put \( ||t|| := \inf_{n \in \mathbb{Z}} |t - n| \) and
\[
(3.23) \quad \eta = \sup \left\{ \gamma \in \mathbb{R}_+ : \liminf_{n \to \infty} n^\gamma ||nt|| = 0 \right\}.
\]
The constant \( \eta \) is called the type of \( t \). If \( \eta \) is finite then \( t \) is called of finite type.

**Lemma 3.20.** Let \( t \) be irrational of type \( \eta \). We then have
\[
\begin{align*}
(1) & \quad \text{For each } k < \eta \text{ and } C > 0 \text{ there exists infinitely many rational numbers } \frac{p}{q} \text{ with } \\
& \quad |t - \frac{p}{q}| \leq \frac{C}{q^{k+1}} \\
(2) & \quad \text{For each } k > \eta \text{ there exists a constant } C > 0 \text{ such that for each } m \in \mathbb{N} \\
& \quad ||mt|| > \frac{C}{m^k} \\
(3) & \quad \text{For each } k > \eta, q \in \mathbb{N} \text{ there exists a constant } C > 0 \text{ such that for each } m, p \in \mathbb{N} \\
& \quad |mt - \frac{p}{q}| > \frac{C}{m^k}
\end{align*}
\]

**Proof.** This follows direct from the definition 3.19 and a small calculation. \( \square \)

We can also estimate the discrepancy of \( \{\{mt\}\}_{m \in \mathbb{N}} \) if \( t \) is of finite type.

**Theorem 3.21.** Let \( x = e^{2\pi it} \) be of finite type \( \eta \) and \( t = \{\{mt\}\}_{m \in \mathbb{N}} \). We then have for each \( \epsilon > 0 \)
\[
(3.24) \quad D_n([0, 1], t) = O \left( n^{-\frac{\eta}{2} + \epsilon} \right)
\]

**Proof.** See [11]. \( \square \)

4. Auxiliary central limit theorem

We prove in this chapter the following auxiliary central limit theorem

**Theorem 4.1.** Let \( \theta > 0 \) be fixed. Let \( \{c_m\}_{m=1}^\infty \) be a sequence of complex numbers with \( a_m = \text{Re}(c_m), b_m = \text{Im}(c_m) \) and
\[
\begin{align*}
(1) & \quad |b_m| \leq 2\pi \text{ and } |a_m| = O(\log(m)) \\
(2) & \quad \frac{1}{n} \sum_{m=1}^n |a_m| = E_a + O(n^{-\delta_a}) \quad \text{for } n \to \infty \text{ and some } \delta_a > 0, \\
(3) & \quad \frac{1}{n} \sum_{m=1}^n |b_m| = E_b + O(n^{-\delta_b}) \quad \text{for } n \to \infty \text{ and some } \delta_b > 0, \\
(4) & \quad \frac{1}{n} \sum_{m=1}^n a_m^2 \to V_a, \quad \frac{1}{n} \sum_{m=1}^n b_m^2 \to V_b, \quad \frac{1}{n} \sum_{m=1}^n a_m b_m \to E_{ab} \quad \text{for } n \to \infty, \\
(5) & \quad \frac{1}{n} \sum_{m=1}^n |a_m|^3 = o(\log^{1/2}(n)), \\
(6) & \quad \text{It exists a } p > \frac{1}{2} \text{ with } \frac{1}{n} \sum_{n/2 < m < n} |a_m|^p = O(1).
\end{align*}
\]
We define \( A_n = A_n(\theta) := \sum_{m=1}^{n} a_m C_m^{(n)} \) and \( B_n = B_n(\theta) := \sum_{m=1}^{n} b_n C_m^{(n)} \).

We then have
\[
\frac{A_n + iB_n - \mathbb{E} [A_n + iB_n]}{\sqrt{\log(n)}} \xrightarrow{d} \mathcal{N}_a + i\mathcal{N}_b
\]

with \( \mathcal{N}_a \) and \( \mathcal{N}_b \) normal distributed random variables with variance \( V_a \) resp. \( V_b \).

The covariance of \( \mathcal{N}_a \) and \( \mathcal{N}_b \) is equal to \( E_{ab} \). The random variables \( \mathcal{N}_a \) and \( \mathcal{N}_b \) are independent if and only if \( E_{ab} = 0 \).

Theorem 4.1 is similar to lemma 3.1 in [9], but there are two important differences:

- We can calculate the covariance between the real and imaginary part and show when they are independent in the limit.
- We consider a more general measure on \( S_n \).

**Proof.** We use in this proof the Feller-coupling (see chapter 3.1). The random variables \( C_m^{(n)}, C_m^{(n+1)} \) and \( Y_m \) are therefore defined on the same space. The idea of the proof is to replace \( C_m^{(n)} \) by \( Y_m \) and to do the computations with \( Y_m \).

We set \( \tilde{A}_n := \sum_{m=1}^{n} a_m Y_m \) and \( \tilde{B}_n := \sum_{m=1}^{n} b_m Y_m \). We have

**Lemma 4.2.** Let \( a_m, b_m, A_n, B_n, \tilde{A}_n \) and \( \tilde{B}_n \) be as above. We then have
\[
\frac{1}{\sqrt{\log(n)}} \mathbb{E} \left[ \tilde{A}_n + i\tilde{B}_n - A_n - iB_n \right] = O(\log^{-1/2}(n)).
\]

The random variables \( \frac{1}{\sqrt{\log(n)}} (A_n + iB_n) \) and \( \frac{1}{\sqrt{\log(n)}} (\tilde{A}_n + i\tilde{B}_n) \) have thus the same asymptotic behavior for \( \theta > 0 \).

We first finish the proof of theorem 4.1 and then prove lemma 4.2. We do this by computing the characteristic function of \( \frac{1}{\sqrt{\log(n)}} (A_n + i\tilde{B}_n) \). We set
\[
\chi^{(n)}(t_a, t_b) = \mathbb{E} \left[ \exp \left( it_a \frac{\tilde{A}_n - \mathbb{E} [\tilde{A}_n]}{\sqrt{\log(n)}} + it_b \frac{\tilde{B}_n - \mathbb{E} [\tilde{B}_n]}{\sqrt{\log(n)}} \right) \right]
\]

We can compute \( \chi^{(n)}(t_a, t_b) \) explicitly since \( \mathbb{E} [e^{itY}] = \exp(\lambda(e^{it} - 1)) \) for \( Y \) a Poisson distributed random variable with \( \mathbb{E} [Y] = \lambda \). We get
\[
\chi^{(n)}(t_a, t_b) = \mathbb{E} \left[ \exp \left( it_a \frac{\tilde{A}_n - \mathbb{E} [\tilde{A}_n]}{\sqrt{\log(n)}} + it_b \frac{\tilde{B}_n - \mathbb{E} [\tilde{B}_n]}{\sqrt{\log(n)}} \right) \right] \exp \left( -it_a \frac{\mathbb{E} [\tilde{A}_n]}{\sqrt{\log(n)}} - it_b \frac{\mathbb{E} [\tilde{B}_n]}{\sqrt{\log(n)}} \right)
= \mathbb{E} \left[ \exp \left( \sum_{m=1}^{n} \frac{it_am + it_mb_m}{\sqrt{\log(n)}} Y_m \right) \right] \exp \left( -it_a \frac{\mathbb{E} [\tilde{A}_n]}{\sqrt{\log(n)}} - it_b \frac{\mathbb{E} [\tilde{B}_n]}{\sqrt{\log(n)}} \right)
= \exp \left( \theta \sum_{m=1}^{n} \frac{e^{(it_am + it_mb_m)/\sqrt{\log(n)}} - 1}{m} \right) \exp \left( -\theta \frac{\mathbb{E} [\tilde{A}_n]}{\sqrt{\log(n)}} (\sum_{m=1}^{n} \frac{it_am + it_mb_m}{m}) \right)
\]
We write as next $e^{it} = \cos(t) + i \sin(t)$ and use the Taylor expansions of $\cos$ and $\sin$. We have
\[ \cos(t) = 1 - \frac{t^2}{2} + \frac{t^4}{24} \sin(\nu) \] with some $\nu \in [0, t]$ and thus $\cos(t) = 1 - \frac{t^2}{2} + O(t^3)$. A similar calculation holds for $\sin$. We get
\[
\cos \left( \frac{t_a a_m}{\sqrt{\log(n)}} \right) = 1 - \frac{t_a^2 a_m^2}{2 \log(n)} + O \left( \frac{t_a^3 a_m^3}{\log^{3/2}(n)} \right),
\]
(4.5)
\[
\sin \left( \frac{t_a a_m}{\sqrt{\log(n)}} \right) = \frac{t_a a_m}{\sqrt{\log(n)}} + O \left( \frac{t_a^3 a_m^3}{\log^{3/2}(n)} \right),
\]
(4.6)
where the big $O$'s are uniformly in $m \in \mathbb{N}$. Similar formulas hold for $b_m$. We use this two identities together and get
\[
\frac{e^{it_a a_m} - e^{-it_b b_m}}{e^{it_b b_m} - e^{-it_a a_m}} - 1 = \frac{1}{\sqrt{\log(n)}} (it_a a_m + it_b b_m) - \frac{1}{\log(n)} \left( \frac{1}{2} a_m^2 t_a^2 + \frac{1}{2} b_m^2 t_b^2 + a_m b_m t_a t_b \right)
\]
(4.7)
\[ + \frac{1}{\log^{3/2}(n)} O \left( |a_m| + |a_m|^2 + |a_m|^3 \right) \]
(4.8)
for $|t_a|, |t_b| \leq K$ with $K$ an arbitrary fixed number. We get
\[
\chi^{(n)}(t_a, t_b) = \exp \left( -\frac{\theta}{2 \log(n)} \sum_{m=1}^{n} \frac{a_m^2 t_a^2 + b_m^2 t_b^2}{m} - t_a t_b \frac{\theta}{\log(n)} \sum_{m=1}^{n} \frac{a_m b_m}{m} \right)
\]
\[ + \frac{\theta}{\log^{1/2}(n)} O \left( \frac{1}{\log(n)} \sum_{m=1}^{n} \frac{|a_m| + |a_m|^2 + |a_m|^3}{m} \right) \).
\]
(4.9)
We apply lemma 3.6 to each summand. The first summand converge by condition 1 to $-\theta(\frac{t_a^2}{2} + \frac{t_b^2}{2})$. The second summand converge by condition 4 to $\theta t_a t_b E_{ab}$. The third summand converge by the conditions 2, 4 and 5 to 0. Therefore
\[
\chi_n(t_a, t_b) \to e^{-\theta\frac{t_a^2}{2} - \frac{t_b^2}{2} + E_{ab} t_a t_b} 
\]
(4.10)
point wise for all $|t_a|, |t_b| \leq K$. Since $K$ was arbitrary, $\chi_n(t_a, t_b)$ converge everywhere. This proves the theorem. \hfill \Box

We now finish the proof by proving lemma 4.2.

Proof of lemma 4.2. We have to distinguish the case $\theta \geq 1$ and $0 < \theta < 1$. We begin with $\theta \geq 1$ and use (3.3) and the conditions 2 and 3 to get
\[
\frac{1}{\sqrt{\log(n)}} \mathbb{E} \left[ \tilde{A}_n + i \tilde{B}_n - A_n - i B_n \right] = \frac{1}{\sqrt{\log(n)}} \mathbb{E} \left[ \sum_{m=1}^{n} (a_m + i b_m)(Y_m - C_m^{(n)}) \right]
\]
\[ \leq \frac{1}{\sqrt{\log(n)}} \left( \frac{\theta(\theta + 1)}{\theta + n} \sum_{m=1}^{n} (|a_m| + |b_m|) \right) = O(\log^{-1/2}(n)).
\]
(4.11)
This proves the lemma for $\theta \geq 1$. 
The case $0 < \theta < 1$ is a little bit more difficult since (3.3) is now weaker. We solve this problem by splitting the sum. We get

\begin{equation}
\frac{1}{\sqrt{\log(n)}} E \left[ |\tilde{A}_n + i\tilde{B}_n - A_n - iB_n| \right] \leq \frac{1}{\sqrt{\log(n)}} E \left[ \sum_{m \leq n/2} (a_n + ib_m)(Y_m - C_m(n)) \right] \\
+ \frac{1}{\sqrt{\log(n)}} E \left[ \sum_{n/2 < m \leq n} (a_m + ib_m)Y_m \right] \\
+ \frac{1}{\sqrt{\log(n)}} E \left[ \sum_{n/2 < m \leq n} (a_m + ib_m)C_m \right]
\end{equation}

Equation (3.3) gives us $E \left[ |C_m(n) - Y_m| \right] \leq \frac{\theta^{(\theta+1)}}{\theta+\theta-\theta} \leq \frac{2\theta^{(\theta+1)}}{n}$ for $1 \leq m \leq n/2$. We thus can use the same computation as in (4.11) to see that the first summand in (4.12) is equal to $O(\log^{-1/2}(n))$. We next look at the second summand in (4.12). We have

\begin{equation}
E \left[ \sum_{n/2 < m \leq n} (a_m + ib_m)Y_m \right] \leq \sum_{n/2 < m \leq n} \theta \left( \frac{|a_m|}{m} + \frac{|b_m|}{m} \right).
\end{equation}

We now use lemma 3.6 and condition (2) and (3) to see that

\begin{equation}
\sum_{n/2 < m \leq n} \frac{|a_m|}{m} = E_a \log(2) + O(n^{-\delta_a}), \quad \sum_{n/2 < m \leq n} \frac{|b_m|}{m} = E_b \log(2) + O(n^{-\delta_b})
\end{equation}

This shows that the second summand in (4.12) is also $O(\log^{-1/2}(n))$. We finally look at the third summand. It is obvious that in a permutation $\sigma \in S_n$ at most one cycle with length greater than $n/2$ can appear. This fact and condition (1) together gives us

\begin{equation}
E \left[ \sum_{n/2 < m \leq n} (a_m + ib_m)C_m \right] = \sum_{n/2 < m \leq n} |a_m + ib_m| P[C_m = 1] \leq \sum_{n/2 < m \leq n} (|a_m| + |b_m|) P[C_m = 1] \\
\leq \sum_{n/2 < m \leq n} |a_m| P[C_m = 1] + 2\pi \left( \sum_{n/2 < m \leq n} P[C_m = 1] \right) \\
= \sum_{n/2 < m \leq n} |a_m| P[C_m = 1] + 2\pi \left( \sum_{n/2 < m \leq n} C_m > 0 \right) \\
\leq 2\pi + \sum_{n/2 < m \leq n} |a_m| P[C_m = 1].
\end{equation}
We have used on the third line that \(\{C_m = 1\} \cap \{C_m = 1\} = \{\}\) for \(m_1, m_2 > n/2\) and \(m_1 \neq m_2\).

If the sequence \((a_m)_{m=1}^{\infty}\) is bounded by a constant \(C\), we can argue in (4.15) as for \(b_m\) to see that

\[
(4.16) \quad \left| \sum_{n/2 < m \leq n} (a_m + ib_m)C_m \right| \leq 2\pi + C
\]

This shows that the third summand in (4.12) is \(O(\log^{-1/2}(n))\) for all \(\theta > 0\) if the sequence \((a_m)_{m=1}^{\infty}\) is bounded.

If the sequence \((a_m)_{m=1}^{\infty}\) is unbounded, we have to be more careful. We first look at \(\mathbb{P}[C_m = 1]\). It follows immediately from (2.5) that

\[
(4.17) \quad \mathbb{P}[C_m(n) = 1] = \mathbb{E}[C_m(n)] = \frac{\theta (n-m-\gamma)}{m(n-\gamma)} \quad \text{for } m > n/2 \text{ with } \gamma = 1 - \theta,
\]

We now need an upper bound for \(\frac{n-m-\gamma}{n-\gamma}\). A simple computation shows

\[
\frac{n-(n-m-\gamma)}{(n-\gamma)} = \frac{n(n-1) \cdots (n-m+1)}{(n-\gamma)(n-\gamma-1) \cdots (n-m-\gamma+1)}
\]

We thus have

\[
\log \left( \frac{n-m-\gamma}{n-\gamma} \right) = \sum_{k=n-m+1}^{n} \log(k) - \sum_{k=n-m+1}^{n} \log(k-\gamma) = \sum_{k=n-m+1}^{n} -\log \left( 1 - \frac{\gamma}{k} \right)
\]

\[
= \sum_{k=n-m+1}^{n} \left( \frac{\gamma}{k} + O \left( \frac{\gamma^2}{k^2} \right) \right) = O(1) + \gamma \sum_{k=n-m+1}^{n} \frac{1}{k}
\]

We now use \(\sum_{m=1}^{n} \frac{1}{m} = \log(n) + C_1 + O \left( \frac{1}{n} \right)\) (see [1, Theorem 3.2]) and distinguish the cases \(m = n\) and \(m < n\). If \(m = n\) then we get immediately that

\[
(4.18) \quad \log \left( \frac{n-m-\gamma}{n-\gamma} \right) \leq \gamma \log(n) + C_2.
\]

If \(m < n\) then

\[
\log \left( \frac{n-m-\gamma}{n-\gamma} \right) = O(1) + \gamma \log(n) - \gamma \log(n-m) + O \left( \frac{1}{n} + \frac{1}{n-m} \right)
\]

\[
= -\gamma \log \left( 1 - \frac{m}{n} \right) + O(1)
\]

since \(n-m > 0\). We put everything together and get

\[
(4.19) \quad \frac{n-m-\gamma}{n-\gamma} \leq \begin{cases} 
C_3(1 - \frac{m}{n})^{-\gamma}, & \text{for } m < n \\
C_4 n^\gamma, & \text{for } m = n.
\end{cases}
\]
We use condition \([\square]\) and the Hölder inequality (see lemma \([3.7]\)) for some \(p, q > 1\), specified in a moment.

\[
\sum_{n/2 < m \leq n} |a_m| \mathbb{P}[C_m = 1] = |a_n| \mathbb{P}[C_n = 1] + \sum_{n/2 < m < n} |a_m| \mathbb{P}[C_m = 1] \\
\leq \frac{O(\log(n))}{n} n^\gamma + \frac{2}{n} \sum_{n/2 < m < n} |a_m| \frac{(n-m-\gamma)}{(n-\gamma)}
\]

\[
\leq \frac{1}{n} \left( \sum_{n/2 < m < n} |a_m|^p \right)^{1/p} \left( \sum_{n/2 < m < n} \left| \frac{(n-m-\gamma)}{(n-\gamma)} \right|^q \right)^{1/q} + O(1)
\]

\[
(4.20) \\
\leq C_3 \left( \frac{1}{n} \sum_{n/2 < m < n} |a_m|^p \right)^{1/p} \left( \frac{1}{n} \sum_{n/2 < m < n} \left( \frac{1 - m}{n} \right)^{\gamma q} \right)^{1/q} + O(1)
\]

The second factor is a Riemann sum for \(\int_1^2 (1-t)^{-\gamma q} \, dt\). If we choose a \(q > 1\) with \(\gamma q < 1\) then the integral exists and one can use theorem 3.16 to see that the second factor converge to this integral. We now check if we can choose \(q\) in such a way that the product is \(O(1)\). We have

\[
\gamma q < 1 \iff (1 - \theta) < \frac{1}{q} \iff (1 - \theta) < 1 - \frac{1}{p} \iff \theta > \frac{1}{p} \iff p > \frac{1}{\theta}.
\]

Condition \([\square]\) now ensures the existence of a \(p > \frac{1}{\theta}\) such that \(\frac{1}{n} \sum_{n/2 < m < n} |a_m|^p\) is \(O(1)\). We get with this \(p\) (and \(q\)) that the product is bounded. This shows that the third summand in (4.12) is \(O(\log^{-1/2}(n))\). \(\square\)

Many assumptions we need in the proof of theorem 4.1 are to handle the case \(\theta < 1\). If one is only interested in the case \(\theta \geq 1\), one can weaken the assumptions. We state this as a corollary.

**Corollary 4.2.1.** Let \(\theta \geq 1\) be fixed. Let \((c_m)_{m=1}^\infty\) be a sequence of complex numbers with \(a_m = \text{Re}(c_m)\), \(b_m = \text{Im}(c_m)\) and

\[
(1') \quad |b_m| \leq 2\pi.
\]

\[
(2') \quad \frac{1}{n} \sum_{m=1}^n |a_m| = o(\log^{1/2}(n)).
\]

\[
(3') \quad \frac{1}{n} \sum_{m=1}^n |b_m| = o(\log^{1/2}(n)).
\]

\[
(4) \quad \frac{1}{n} \sum_{m=1}^n a_m^2 \rightarrow V_a, \quad \frac{1}{n} \sum_{m=1}^n b_m^2 \rightarrow V_b, \quad \frac{1}{n} \sum_{m=1}^n a_m b_m \rightarrow E_{ab} \text{ for } n \rightarrow \infty.
\]

\[
(5) \quad \frac{1}{n} \sum_{m=1}^n |a_m|^3 = o(\log^{1/2}(n)).
\]

Let \(A_n\) and \(B_n\) be as in theorem 4.1. Then

\[
(4.21) \quad \frac{A_n + iB_n - \mathbb{E}_\theta [A_n + iB_n]}{\sqrt{\log(n)}} \overset{d}{\rightarrow} \mathcal{N}_a + i\mathcal{N}_b
\]

with \(\mathcal{N}_a\) and \(\mathcal{N}_b\) normal distributed random variables with variance \(V_a\) resp. \(V_b\).

The covariance of \(\mathcal{N}_a\) and \(\mathcal{N}_b\) is equal to \(E_{ab}\). The random variables \(\mathcal{N}_a\) and \(\mathcal{N}_b\) are independent if and only if \(E_{ab} = 0\).
5. Proof of the main theorem 2.8

We now are ready to prove theorem 2.8. We recommend to read first chapter 3.3 and chapter 3.4 before reading this proof.

Proof of theorem 2.8: We have by definition $w^n(f)(x) = \sum_{m=1}^{n} C_m^n \log(f(x^m))$. We now apply theorem 4.1 with $c_m := \log(f(x^m))$. To do this, we have to show that the conditions (1) - (6) are fulfilled.

We now write $x = e^{2\pi i t}$, $t_m := \{mt\}$, $t = (t_m)_{m=1}^{\infty}$ and $a(s) := \log|f(e^{2\pi i s})|$, $h(s) := |\log|f(e^{2\pi i s})||$ and $b(s) := \arg(f(e^{2\pi i s}))$. We have to distinguish several cases:

Case 1.1: $x$ not a root of unity, $0 \notin f(S^1)$

Condition (1) is trivially fulfilled since $a_m$ is bounded in this case and $b_m \leq 2\pi$ by definition of $w^n(f)$. We next look at condition (2). We have

$$\frac{1}{n} \sum_{m=1}^{n} |a_m| = \frac{1}{n} \sum_{m=1}^{n} |\text{Re}(c_m)| = \frac{1}{n} \sum_{m=1}^{n} |\log|f(x^m)|| = \frac{1}{n} \sum_{m=1}^{n} h(t_m).$$

The function $h(s)$ is in this case a real analytic function on $[0,1]$ and we therefore can apply theorem 3.9 to see that the last expression converges to $\int_{0}^{1} h(s) \, ds$. We need in (2) also the rate of convergence. We thus use theorem 3.15 instead of theorem 3.9. We therefore need an upper bound for $D_n^e([0,1], t)$. Since $x$ is of finite type, we can use theorem 3.21 to get the desired upper bound of the form $O(n^{-\delta_k})$. It follows with the same argument that

$$\frac{1}{n} \sum_{m=1}^{n} |a_m|^p \to \int_{0}^{1} h^p(s) \, ds \text{ for each } 1 \leq p < \infty.$$

This shows that conditions (1), (3), (4) and the first part of condition (5) are fulfilled.

We next look at $b_m = \text{Im} \left( \log(f(x^m)) \right) = \arg(f(x^m)) = b(t_m)$. Since $f$ is real analytic, one can use function theory to show that there exists a finite set $D \subset [0,1]$ such that $b(s)$ is real analytic in $[0,1] \setminus D$ and the limits $\lim_{s \to \tau_m} \arg(f(e^{2\pi i s}))$ and $\lim_{s \to \imath_m} \arg(f(e^{2\pi i s}))$ exists for all $s \in [0,1]$. We omit here the details since this is a standard argument.

The idea is now to split $[0,1]$ into subintervals such that $b(s)$ is real analytic in each and then to restrict the sequence $t$ to each such subinterval. Lemma 3.18 shows that the restricted sequences are still uniformly distributed and we are therefore allowed to use theorem 3.9 and theorem 3.15 on each subinterval separately. We get

$$\frac{1}{n} \sum_{m=1}^{n} |b_m| = \int_{0}^{1} |b(s)| \, ds + O(n^{-\delta_k}) \quad \text{and} \quad \frac{1}{n} \sum_{m=1}^{n} b_m^2 \to \int_{0}^{1} (b(s))^2 \, ds$$

and

$$\frac{1}{n} \sum_{m=1}^{n} a_m b_m = \frac{1}{n} \sum_{m=1}^{n} a(t_m) b(t_m) \to \int_{0}^{1} a(s) b(s) \, ds = \int_{0}^{1} \text{Re} \left( \log(f(e^{2\pi i s})) \right) \text{Im} \left( \log(f(e^{2\pi i s})) \right) \, ds$$

$$= \frac{1}{2} \text{Im} \left( \int_{0}^{1} \log^2(f(e^{2\pi i s})) \, ds \right)$$
The second equation gives the desired expression for the correlation mentioned in theorem 2.8. This shows that condition 3 and the rest of condition 4 are fulfilled.

Case 1.2.1 \( x \) not a root of unity and 1 is the only zero of \( f \).
A simple calculation shows that \( h(s) \sim C_1 \log(s) \) and \( \frac{d}{ds} h(s) \sim C_2 \frac{1}{s} \) for \( s \to 0 \). Similar for \( s \to 1 \). We therefore can not use anymore theorem 3.15. The idea is to replace theorem 3.15 by theorem 3.16. To apply theorem 3.16, we have to know the behavior of \( D_n([0, 1], t) \) for \( n \to \infty \) and to choose a “good” \( \delta > 0 \) with \( \delta < t_m < 1 - \delta \).
This is the point where we need that \( x \) is of finite type. The behavior of \( D_n([0, 1], t) \) then follows immediately from theorem 3.21. We have
\[
D_n([0, 1], t) = O(n^{-\alpha}) \quad \text{for some } \alpha > 0
\]
We now have to choose \( \delta \). It follows from lemma 3.20 (and \( x \) of finite type) that there exists constants \( C_3 > 0, \beta > 0 \) such that
\[
|m x - p| > \frac{C_3}{m^\beta} \text{ for all } m, p \in \mathbb{N}.
\]
We set \( \delta = \delta(n) = \frac{C_3}{n^\beta} \). This \( \delta \) fulfills the desired property \( \delta < t_m < 1 - \delta \) for \( 1 \leq m \leq n \). We get
\[
(5.1) \quad \frac{1}{n} \sum_{m=1}^{n} h(t_m) - \int_{0}^{1} h(t) \, dt \leq \int_{0}^{\delta} h(s) \, ds + \int_{1-\delta}^{1} h(s) \, ds
\]
\[
+ D_n([0, 1], t) \int_{\delta}^{1-\delta} \left| \frac{d}{ds} h(s) \right| \, ds + (\delta h(\delta) + \delta h(1-\delta))
\]
\[
\leq O(n^{-\beta} \log(n^3)) + O(n^{-\alpha} \log(n^3)) + O(n^{-\beta} \log(n^3))
\]
\[
= O(n^{-\delta_n}) \text{ for some } \delta_n > 0
\]
This shows that condition 2 is fulfilled. Condition 4 also fulfills immediately with this choice of \( \delta \). A simple calculation shows that
\[
h^2(s) \sim C_4 \log^2(s), \quad \frac{d}{ds} h^2(s) \sim C_5 \frac{\log(s)}{s}
\]
\[
h^3(s) \sim C_6 \log^3(s), \quad \frac{d}{ds} h^3(s) \sim C_7 \frac{\log^2(s)}{s}
\]
\[
h^p(s) \sim C_8 \log^p(s), \quad \frac{d}{ds} h^p(s) \sim C_9 \frac{\log^{p-1}(s)}{s}
\]
It is now easy to see that one can use the calculations in (5.1) also for \( h^2, h^3 \) and \( h^p \). This proves conditions 5, 6, and the first part of condition 4.
The argumentation for \( b_m \) is as above.

Case 1.2.2 \( x \) not a root of unity and all zeros of \( f \) roots of unity.
Choose a \( q \in \mathbb{N} \) fix such that all zeros of \( f \) can be written as \( \exp(2\pi i \frac{p}{q}) \) with \( p \in \mathbb{N} \). This is possible since by assumption all zeros of \( f \) are roots of unity and \( f \) is real analytic.
We set \( I_p := [\frac{p-1}{q}, \frac{p}{q}] \) for \( 1 \leq p \leq q \) and apply theorem 3.16 separately to each \( I_p \) for the sequences \( y^{(p)} = x \cap I_p \). We know from lemma 3.18 that \( D_n(I_p, y^{(p)}) = O(n^{-\alpha}) \) since \( D_n([0, 1], x) = O(n^{-\alpha}) \).
We can choose as above with lemma 3.20 a \( \delta = \frac{C_3}{n^\beta} \) with \( \frac{p-1}{q} + \delta < y_m^{(p)} < \frac{p}{q} - \delta \) for \( 1 \leq m \leq n \).
This shows that the calculations are similar to the calculations in Case 1.2.1.

Case 2: \( x \) a root of unity of order \( p \) and \( f(x^m) \neq 0 \) for all \( 1 \leq m \leq p \).

We have

\[
\frac{1}{n} \sum_{m=1}^{n} |a_m| = \frac{1}{n} \sum_{m=1}^{p} h(t_m) = \sum_{k=1}^{p} \left( \frac{1}{\log(n)} \sum_{j=0}^{\left\lceil \frac{n-k}{p} \right\rceil} \frac{1}{jp+k} \right) h(t_{jp+k})
\]

(5.2)

It is easy to see that

\[
\lim_{n \to \infty} \frac{1}{\log(n)} \sum_{j=0}^{\left\lceil \frac{n-k}{p} \right\rceil} \frac{1}{jp+k} = \lim_{n \to \infty} \frac{1}{\log(n/p) + \log(p)} \sum_{j=0}^{\left\lfloor \frac{n}{p} \right\rfloor} \frac{1}{jp} = \frac{1}{p}
\]

This shows the desired convergence. To get the rate of convergence, one has to use \( \sum_{j=1}^{n} \frac{1}{j} = \log(n) + C_{10} + O\left(\frac{1}{n}\right) \) (see [1]). We therefore have proven that (2) is fulfilled.

The other calculations are similar. We therefore omit them.

We have until now proven that

\[
\frac{w^n(f) - E[w^n(f)]}{\sqrt{\log(n)}} \xrightarrow{d} \mathcal{N}_a + i\mathcal{N}_b
\]

in all cases mentioned in theorem 2.8 inclusive the calculation of the correlation.

To complete the proof, we have to show that \( \mathbb{E}[w^n(f)] \sqrt{\log(n)} - \theta \sqrt{\log(n)} m(f) \to 0 \). We define as in the proof of theorem 4.1

\[
\bar{w}^n(f)(x) := \sum_{m=1}^{n} \log\left(f(x^m)\right) Y_m
\]

We know from lemma 4.2 that \( w^n(f) \) and \( \bar{w}^n(f) \) have the same asymptotic behavior. It is therefore enough to prove \( \mathbb{E}[\bar{w}^n(f)] \sqrt{\log(n)} - \theta \sqrt{\log(n)} m(f) \to 0 \). We first look at the case \( x \) not a root of unity. We get with lemma 3.6 and (5.1)

\[
\mathbb{E}[\bar{w}^n(f)(x)] = \theta \sum_{m=1}^{n} \frac{\log\left(f(x^m)\right)}{m} = \log(n) \theta \int_0^1 \log\left(f(e^{2\pi is})\right) ds + \text{Const.} + O(1/n).
\]

We have by definition that \( m(f) = \int_0^1 \log\left(f(e^{2\pi is})\right) ds \) and thus

\[
\frac{1}{\sqrt{\log(n)}} \sum_{m=1}^{n} \frac{\log\left(f(x^m)\right)}{m} = \sqrt{\log(n)} m(f) + O\left(\frac{1}{\sqrt{\log(n)}}\right)
\]

(5.6)

In this case \( x \) a root of unity, one has to replace (5.1) as before by \( \sum_{m=1}^{n} \frac{1}{m} = \log(n) + C_{10} + O\left(\frac{1}{n}\right) \).

We omit the details since these calculations are similar as above. \( \square \)
6. Estimation of the Wasserstein distance

We use in this section Stein’s method to estimate the convergence rate of the probability measures in theorem 2.8 (see theorem 6.2 and theorem 6.8).

We give at the begin of section 6.2 a very short overview to the idea of Stein’s method. An introduction to Stein’s method can be found in [5].

Unfortunately we have to distinguish the cases $X$ a random variable with values in $\mathbb{R}^1$ and $X$ a random variable with values in $\mathbb{R}^d$. We thus look in section 6.1 first at the real and the imaginary part of $w^n(f)$ separately and prove there an upper bound for the Wasserstein distance. We then look in section 6.2 at the $2$-dimensional case and prove an upper bound for a weak Wasserstein distance.

6.1. The real and the imaginary part separately. We first introduce

**Definition 6.1.** Let $X_1, X_2$ be real valued random variables not necessarily defined on the same space. The Wasserstein distance $d_W$ between $X_1$ and $X_2$ is then defined as

$$d_W(X_1, X_2) := \sup_{g \in \mathcal{G}} |\mathbb{E}[g(X_1)] - \mathbb{E}[g(X_2)]|$$

with

$$\mathcal{G} := \left\{ g \in C^1(\mathbb{R}, \mathbb{R}); \sup_{x_1 \neq x_2} \left| \frac{g(x_1) - g(x_2)}{x_1 - x_2} \right| \leq 1 \right\}.$$  

We now show

**Theorem 6.2.** Let $\theta > 0$, $x$ a root of unity or $x$ not a root of unity and of finite type. Let $N_a, N_b$ be as in theorem 2.8. We then have

$$d_W\left(N_a, \text{Re} \left( \frac{w^n(f)(x)}{\sqrt{\log(n)}} - \theta \sqrt{\log(n)} m(f)(x) \right) \right) = O\left(\log^{-1/2}(n)\right),$$

$$d_W\left(N_b, \text{Im} \left( \frac{w^n(f)(x)}{\sqrt{\log(n)}} - \theta \sqrt{\log(n)} m(f)(x) \right) \right) = O\left(\log^{-1/2}(n)\right).$$

**Proof.** We assume in this proof that $x$ is not a root of unity and of finite type. The argumentation for $x$ a root of unity is the same. There are indeed only some minor differences in certain formulas. We use the notation

$$c_m = \log(f(\alpha^m)), \quad a_m = \text{Re}(c_m), \quad b_m = \text{Re}(c_m), \quad a(t) = \log|f(e^{2\pi it})|, \quad b(t) = \arg(f(e^{2\pi it})).$$

We define $\bar{w}^n(f)$ as in (5.4). It follows from lemma 4.2 that

$$\mathbb{E} \left[ \frac{w^n(f)(x)}{\sqrt{\log(n)}} - \frac{\bar{w}^n(f)(x)}{\sqrt{\log(n)}} \right] = O\left(\log^{-1/2}(n)\right).$$

Since every $g \in \mathcal{G}$ Lipschitz continuous, we can replace $w^n(f)$ by $\bar{w}^n(f)$. We now use (5.6) to replace

$$\theta \sqrt{\log(n)} m(f)(x) \quad \text{by} \quad \frac{\theta}{\sqrt{\log(n)}} \sum_{m=1}^{n} \frac{c_m}{m}. $$
We next look at $N_a, N_b$. We know from theorem 4.1, the computations in section 5 and lemma 3.5 that
\[ V_a = \text{Var}(N_a) = \int_0^1 a^2(t) \, dt = \frac{1}{\log(n)} \sum_{m=1}^n \frac{a_m^2}{m} + O(\log^{-1}(n)). \]

We write
\[ N_a \overset{d}{=} \mathcal{N}(0, \frac{1}{\log(n)} \sum_{m=1}^n a_m^2). \]

If $N$ is a random variable with $N \sim \mathcal{N}(0, O(\log(n)))$, then
\[ \mathbb{E}[|N|] = O\left(\frac{1}{\sqrt{\log(n)}}\right). \]

This shows that we can replace $V_a$ by $\frac{1}{\log(n)} \sum_{m=1}^n a_m^2$. Similarly for the imaginary part. We put everything together and see that it is sufficient to show
\[ d_W\left(\mathcal{N}\left(0, \frac{1}{\log(n)} \sum_{m=1}^n a_m^2\right), \mathcal{N}\left(0, \frac{1}{\log(n)} \sum_{m=1}^n b_m^2\right)\right) = O(\log^{-1/2}(n)). \]

This situation was already considered in [5] using Stein’s method. We have

**Theorem 6.3** ([5, theorems 3.1 and 3.2]). Let $\xi_1, \ldots, \xi_n$ be independent random variables with
\[ \mathbb{E}[\xi_m] = 0, \quad \mathbb{E}\left[\sum_{m=1}^n \xi_m^2\right] = V \quad \text{and} \quad \mathbb{E}[|\xi_m^3|] < \infty. \]

Then
\[ d_W\left(\sum_{m=1}^n \xi_m, \mathcal{N}(0, V)\right) \leq 3V^{3/2} \sum_{m=1}^n \mathbb{E}[|\xi_m^3|]. \]

We now set $\xi_m = \frac{1}{\sqrt{\log(n)}} a_m \left(Y_m - \frac{1}{m}\right)$. A simple computation shows that $\mathbb{E}[|Y_m - 1/m|^3] = 1/m + O(1/m^2)$. We get
\[ \sum_{m=1}^n \mathbb{E}[|\xi_m^3|] \leq \text{Const.} \cdot \frac{1}{\log(n)} \left(\frac{1}{\log(n)} \sum_{m=1}^n \frac{|a_m^3|}{m}\right) = O(\log^{-1/2}(n)). \]

The argumentation for the imaginary part is similar. This proves the theorem. □
6.2. **The two dimensional case.** In this section we illustrate shortly the idea of Stein’s method and then apply it to \(w^n(f)\) to get an upper bound for a weak Wasserstein distance.

The idea of Stein’s method is to find for a given function \(g\) and a given random variable \(Z\) a "good" function \(\tilde{g}\) only depending on \(g\) and \(Z\) such that

\[
\mathbb{E}[\tilde{g}(X)] = \mathbb{E}[g(X)] - \mathbb{E}[g(Z)]
\]

for all random variables \(X\). This is of course only useful if we can find a \(\tilde{g}\) with good properties. This is surprisingly often the case. We have for example for \(N \sim \mathcal{N}(0, 1)\) and \(g \in C^1(\mathbb{R}, \mathbb{R})\)

\[
\mathbb{E}[f'(g)(X) - Xf(g)(X)] = \mathbb{E}[g(X)] - \mathbb{E}[g(N)]
\]

with

\[
f(g)(x) := e^{x^2/2} \int_{-\infty}^{x} (g(t) - \mathbb{E}[g](N)) e^{-t^2/2} \, dt.
\]

We now take a closer look at \(N_a + iN_b\). We set

\[
\Sigma := \begin{pmatrix} \text{Cov}(N_a, N_a) & \text{Cov}(N_a, N_b) \\ \text{Cov}(N_b, N_a) & \text{Cov}(N_b, N_b) \end{pmatrix} = \begin{pmatrix} V_a & E_{ab} \\ E_{ab} & V_b \end{pmatrix}.
\]

It follows immediately from (4.10) that we can write the characteristic function of \(N_a + iN_b\) as

\[
\exp \left( -\frac{1}{2} \langle \mathbf{t}, \mathbf{t} \rangle \Sigma \right),
\]

We will see in lemma 6.6 that \(\Sigma\) is always non negative definite. Thus there exists a matrix \(S\) with \(S^T S = \Sigma\). We set

\[
\begin{pmatrix} M_a \\ M_b \end{pmatrix} := S \begin{pmatrix} N_a \\ N_b \end{pmatrix}.
\]

It follows directly from (6.19) that \(M_a\) and \(M_b\) are two independent normal distributed random variables. A random variable \(N_S := N_a + iN_b\) with this property is called bivariate normal distributed. A way to apply Stein’s method to bivariate and to multivariate normal distributed random variables can be found in [12]. We have for instance

**Lemma 6.4 ([12] lemma 1).** Let \(g : \mathbb{R}^2 \to \mathbb{R}\) be a smooth function and \(N_S\) be a bivariate normal distributed random variable. We set

\[
Ug(x) := \int_0^1 \frac{1}{2t} \left( \mathbb{E} \left[ g(\sqrt{t}x + \sqrt{1- tN_S}) \right] - \mathbb{E} \left[ g(N_S) \right] \right) \, dt.
\]

We then have

\[
\mathbb{E} \left[ \langle X, \nabla(Ug)(X) \rangle - \langle \text{Hess}(Ug)(X), \Sigma \rangle_{H.S.} \right] = \mathbb{E} \left[ g(X) \right] - \mathbb{E} \left[ g(N_S) \right]
\]

with \(\langle \cdot, \cdot \rangle\) the standard inner product on \(\mathbb{R}^2\) and \((M_1, M_2)_{H.S.} = \text{Tr}(M_1 M_2^T)\).

We need here some bounds for the derivatives of the function \(Ug\). These bounds can also be found in [12] and are

**Lemma 6.5 ([12] lemma 2).** Let \(g : \mathbb{R}^d \to \mathbb{R}\) be a smooth function and \(Ug\) as in lemma 6.4. We set for \(k \in \mathbb{N}\)

\[
M_k(g) := \sup_{u \in \mathbb{R}} \left| \frac{\partial^k}{\partial u_{i_1} \cdots \partial u_{i_k}} g(u) \right|.
\]

\[
\text{for } 1 \leq i_1, \ldots, i_k \leq 2.
\]
If Σ is positive definite, then
\[(6.24) \quad M_3(Ug) \leq C_{11}M_2(g)\]
with $C_{11}$ only depending on Σ.

There exists also upper bounds for $M_k(Ug)$ if Σ is non negative definite, but we do not need them here. The reason is the following lemma.

**Lemma 6.6.** The matrix Σ is always non negative definite.
The matrix Σ is singular if and only if $\log |f(x_m)| \neq \text{arg} (f(x_m))$ for only finitely many $m$.

**Proof.** We prove this lemma only for $\Sigma$. The matrix $\Sigma$ is non negative definite if and only if
\[(6.25) \quad V_a = \int_0^1 a^2(t) \, dt, \quad E_{ab} = \int_0^1 a(t)b(t) \, dt, \quad V_b = \int_0^1 b^2(t) \, dt.\]
A direct computation shows that the eigenvalues of Σ have the form
\[(6.26) \quad \frac{1}{2} \left( \int_0^1 a^2(t) \, dt + \int_0^1 b^2(t) \, dt \pm \sqrt{\left( \int_0^1 a^2(t) \, dt - \int_0^1 b^2(t) \, dt \right)^2 + 4 \left( \int_0^1 a(t)b(t) \, dt \right)^2} \right).\]
The matrix Σ is non negative definite if and only if
\[(6.27) \quad \left( \int_0^1 a^2(t) \, dt + \int_0^1 b^2(t) \, dt \right)^2 \geq \left( \int_0^1 a^2(t) \, dt - \int_0^1 b^2(t) \, dt \right)^2 + 4 \left( \int_0^1 a(t)b(t) \, dt \right)^2.
This is equivalent to
\[(6.28) \quad \left( \int_0^1 a^2(t) \, dt \right) \left( \int_0^1 b^2(t) \, dt \right) \geq \left( \int_0^1 a(t)b(t) \, dt \right)^2.
But equation (6.28) is the Schwarz inequality for $L^2$. This shows that Σ is always non negative definite and that Σ is singular if and only if $a(t) = b(t)$ for almost all $t \in [0, 1]$.

We now show Σ is singular if and only if $a_m \neq b_m$ for only finitely many $m$.
If Σ is singular then $a(t) = b(t)$ for all common continuity points. The functions $a(t)$ and $b(t)$ have only finitely many discontinuity points and all values of $x_m$ are different since $x$ is not a root of unity. We thus can have $\log |f(x_m)| \neq \text{arg} (f(x_m))$ for only finitely many $m$.
The other direction follows immediately from the fact that the sequence $(x_m)_{m \in \mathbb{N}}$ is dense in $S^1$. □

We are now ready to introduce the weak Wasserstein distance and to state the main theorem of this subsection.

**Definition 6.7.** Let $X_1, X_2$ be random variables with values in $\mathbb{R}^d$, not necessarily defined on the same space. The weak Wasserstein distance $d_{wW}$ between $X_1$ and $X_2$ is then defined as
\[(6.29) \quad d_{wW}(X_1, X_2) := \sup_{g \in \mathcal{G}} \mathbb{E} |g(X_1)| - \mathbb{E} |g(X_2)|\]
with
\[(6.30) \quad \mathcal{G} := \{ g \in C^\infty(\mathbb{R}^d, \mathbb{R}); M_1(g) \leq 1 \text{ and } M_2(g) \leq 1 \}.

Theorem 6.8. Let \( \theta > 0 \), \( x \) a root of unity or \( x \) not a root of unity and of finite type. Let \( \mathcal{N}_\Sigma = \mathcal{N}_a + i\mathcal{N}_b \) be as in theorem 2.8. We then have

\[
\text{d}_{wW} \left( \mathcal{N}_\Sigma, \frac{w^n(f)(x)}{\sqrt{\log(n)}} - \theta \sqrt{\log(n)} m(f)(x) \right) = O \left( \log^{-1/2}(n) \right).
\]

This theorem is an alternative proof of theorem 2.8 since the weak Wasserstein distance is a metric on the space of probability measures.

Proof. Let \( g \) be given with \( M_1(g) \leq 1, M_2(g) \leq 1 \). We have to distinguish the cases \( \Sigma \) singular and \( \Sigma \) regular.

We start with the singular case. We know from lemma 6.6 that \( \Sigma \) is singular if and only if \( a_m \neq b_m \) for only finitely many \( m \). We thus have

\[
\mathbb{E} \left[ g \left( \frac{1}{\sqrt{\log(n)}} \sum_{m=1}^{n} c_m Y_m \right) \right] = \mathbb{E} \left[ \hat{g} \left( \frac{1}{\sqrt{\log(n)}} \sum_{m=1}^{n} \tilde{a}_m Y_m \right) \right] + O \left( \log^{-1/2}(n) \right)
\]

with \( \hat{g}(t) := g((1 + i)t) \). This shows that we can argue as in the 1-dimensional case.

We now come to \( \Sigma \) regular. One can use the same argumentation as in the proof of lemma 6.2 to see that it is enough to show

\[
\text{d}_{wW} \left( \mathcal{N}_\Sigma, \frac{1}{\sqrt{\log(n)}} \sum_{m=1}^{n} c_m \left( Y_m - \frac{1}{m} \right) \right)
\]

with \( \mathcal{N}_\Sigma \) a bivariate normal distribution with

\[
\Sigma = \begin{pmatrix} \tilde{V}_a & \tilde{E}_{ab} \\ \tilde{E}_{ab} & \tilde{V}_b \end{pmatrix} = \begin{pmatrix} \sum_{m=1}^{n} \tilde{a}_m^2 \frac{2}{m} & \sum_{m=1}^{n} \tilde{a}_m \tilde{b}_m \frac{2}{m} \\ \sum_{m=1}^{n} \tilde{a}_m \tilde{b}_m \frac{2}{m} & \sum_{m=1}^{n} \tilde{b}_m^2 \frac{2}{m} \end{pmatrix}
\]

and

\[
\tilde{a}_m = \frac{a_m}{\sqrt{\log(n)}}, \quad \tilde{b}_m = \frac{b_m}{\sqrt{\log(n)}}, \quad \text{and} \quad \tilde{c}_m = \frac{c_m}{\sqrt{\log(n)}}.
\]

We now use (6.22) to give the desired upper bound. Let \( U_g \) be as in (6.21). We use the notation

\[
\nabla(U_g) = \begin{pmatrix} g_a \\ g_b \end{pmatrix}, \quad \text{Hess}(U_g) = \begin{pmatrix} g_{aa} & g_{ab} \\ g_{ab} & g_{bb} \end{pmatrix}, \quad \text{and} \quad X := \sum_{m=1}^{n} \tilde{c}_m \left( Y_m - \frac{1}{m} \right).
\]

We also introduce

\[
X_a := \sum_{m=1}^{n} \tilde{a}_m \left( Y_m - \frac{1}{m} \right), \quad X_b = \sum_{k \neq m}^{1 \leq k \leq n} \tilde{b}_m \left( Y_k - \frac{1}{k} \right), \quad X_m = \sum_{k \neq m}^{1 \leq k \leq n} \tilde{c}_k \left( Y_k - \frac{1}{k} \right).
\]
We now identify $C$ with $\mathbb{R}^2$ via $a + ib = (\Re a, \Im b)$. We first look at the summand $\mathbb{E}[\langle X, \nabla(Ug)(X) \rangle] = \mathbb{E}[X_a g_a(X)] + \mathbb{E}[X_b g_b(X)]$. We have

$$
\mathbb{E}[X_a g_a(X)] = \sum_{m=1}^{n} \bar{a}_m \mathbb{E} \left[ (Y_m - \frac{1}{m}) g_a(X) \right] = \sum_{m=1}^{n} \bar{a}_m \mathbb{E} \left[ (Y_m - \frac{1}{m}) (g_a(X) - g_a(X_m)) \right]
$$

$$
= \sum_{m=1}^{n} \bar{a}_m \mathbb{E} \left[ (Y_m - \frac{1}{m}) \int_{0}^{Y_m} \left\langle \nabla g_a(X_m + t\bar{c}_m), \left( \frac{\bar{a}_m}{\bar{b}_m} \right) \right\rangle \, dt \right]
$$

$$
= \sum_{m=1}^{n} \bar{a}_m \mathbb{E} \left[ (Y_m - \frac{1}{m}) \int_{0}^{\infty} \left\langle \nabla g_a(X_m + t\bar{c}_m), \left( \frac{\bar{a}_m}{\bar{b}_m} \right) \right\rangle \int_{0}^{\infty} \mathbb{1}_{\{0 \leq t \leq Y_m\}}(t) \, dt \right]
$$

$$
= \sum_{m=1}^{n} \mathbb{E} \left[ \int_{0}^{\infty} \left( \bar{a}_m^2 g_{aa}(X_m + t\bar{c}_m) + \bar{a}_m \bar{b}_m g_{ab}(X_m + t\bar{c}_m) \right) K_m(t) \, dt \right]
$$

(6.38)

with

$$
K_m(t) = \mathbb{E} \left[ (Y_m - \frac{1}{m}) \mathbb{1}_{\{0 \leq t \leq Y_m\}} \right].
$$

We have used for the second equality that $\mathbb{E}[Y_m - 1/m] = 0$ and that $X_m$ is independent of $Y_m$. We have of course also to justify the existence of the integrals, but this follows immediately from $M_1(g) \leq 1$, $M_2(g) \leq 1$ and lemma [6.5]. We next look at $\mathbb{E}[\langle \text{Hess}(Ug)(X), \Sigma \rangle_{H.S.}] = \bar{V}_a \mathbb{E}[g_{aa}(X)] + 2\bar{E}_{ab} \mathbb{E}[g_{ab}(X)] + \bar{V}_b \mathbb{E}[g_{bb}(X)]$. A direct computation shows that

$$
\int_{0}^{\infty} K_m(t) \, dt = \frac{1}{m} \quad \text{and} \quad \int_{0}^{\infty} tK_m(t) \, dt = \frac{1}{m} + O \left( \frac{1}{m^2} \right)
$$

with $O(\cdot)$ independent of $n$ and $\Sigma$. We thus have

$$
\mathbb{E}[\bar{V}_a g_{aa}(X)] = \mathbb{E} \left[ g_{aa}(X) \sum_{m=1}^{n} \bar{a}_m^2 \frac{1}{m} \right] = \sum_{m=1}^{n} \mathbb{E} \left[ \bar{a}_m^2 \int_{0}^{\infty} g_{aa}(X) K_m(t) \, dt \right]
$$

(6.41)

We combine (6.38) and (6.41) and get

$$
\mathbb{E}[\langle X, \nabla(Ug)(X) \rangle] - \mathbb{E}[\langle \text{Hess}(Ug)(X), \Sigma \rangle_{H.S.}]
$$

$$
= \sum_{m=1}^{n} \mathbb{E} \left[ \int_{0}^{\infty} \bar{a}_m^2 \left( g_{aa}(X_m + \bar{c}_m t) - g_{aa}(X_m + \bar{c}_m Y_m) \right) K_m(t) \, dt \right]
$$

$$
+ \sum_{m=1}^{n} \mathbb{E} \left[ \int_{0}^{\infty} \bar{a}_m \bar{b}_m \left( g_{ab}(X_m + \bar{c}_m t) - g_{ab}(X_m + \bar{c}_m Y_m) \right) K_m(t) \, dt \right]
$$

$$
+ \sum_{m=1}^{n} \mathbb{E} \left[ \int_{0}^{\infty} \bar{b}_m^2 \left( g_{bb}(X_m + \bar{c}_m t) - g_{bb}(X_m + \bar{c}_m Y_m) \right) K_m(t) \, dt \right]
$$

(6.42)
We now use (6.40) and lemma 6.5 to get
\[
\left| \mathbb{E}\left[ \int_0^\infty \left( g_{aa}(X_m + \tilde{c}_m t) - g_{aa}(X_m + \tilde{c}_m Y_m) \right) K_m(t) \, dt \right] \right| \leq \mathbb{E}\left[ \int_0^\infty M_3(Ug) |\tilde{c}_m| (t + Y_m) K_m(t) \, dt \right] \\
\leq M_2(g) |\tilde{c}_m| \int_0^\infty (t + m) K_m(t) \, dt \\
\leq \frac{C_{11}}{\sqrt{\log(n)}} M_2(g) (|a_m| + |b_m|) \frac{1}{m}.
\]

(6.43)

Thus
\[
\mathbb{E}[\langle X, \nabla (Ug)(X) \rangle] - \mathbb{E}[\langle \text{Hess}(Ug)(X), \Sigma \rangle_{\text{H.S.}}] \leq \frac{C_{13}}{\log^{3/2}(n)} \left( \sum_{m=1}^n |a_m^3| + |a_m^2 b_m| + |a_m b_m^2| + |b_m^3| \right)
\]

(6.44)

It follows with a computation similar to computation in (5.1) that the last expression is \(O\left(\log^{-1/2}(n)\right)\).

This proves the theorem. \(\Box\)

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References


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