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ROOTS OF THE DERIVATIVE OF THE RIEMANN ZETA FUNCTION
AND OF CHARACTERISTIC POLYNOMIALS

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Abstract. We investigate the horizontal distribution of zeros of the derivative of the Riemann zeta function and compare this to the radial distribution of zeros of the derivative of the characteristic polynomial of a random unitary matrix. Both cases show a surprising bimodal distribution which has yet to be explained. We show by example that the bimodality is a general phenomenon. For the unitary matrix case we prove a conjecture of Mezzadri concerning the leading order behavior, and we show that the same follows from the random matrix conjectures for the zeros of the zeta function.

1. Introduction

The zeros of the derivative $\zeta'(s)$ of the Riemann zeta-function are intimately connected with the behavior of the zeros of $\zeta(s)$ itself. Indeed, a theorem by Speiser [19] states that the Riemann Hypothesis (RH) is equivalent to $\zeta'(s)$ having no zeros to the left of the critical line. Thus, understanding of the properties the zeros of $\zeta'(s)$ can provide important tools and insight into the study of RH. After Speiser’s article this idea was explored by Berndt [1] and Spira [20], but not much progress was achieved until the work of Levinson and Montgomery [11], who proved a quantitative refinement of Speiser’s theorem. They showed that $\zeta(s)$ and $\zeta'(s)$ have essentially the same number of zeros to the left of the critical line $\sigma = \text{Re}(s) = \frac{1}{2}$, and proved that as $T \to \infty$, where $T$ is the height on the critical line, a positive proportion of the zeros of $\zeta'(s)$ lie in the region

$$\sigma < \frac{1}{2} + (1 + \epsilon) \frac{\log \log T}{\log T}, \quad \epsilon > 0. \quad (1.1)$$

Consider the group of unitary matrices $U(N)$ with probably distribution given by Haar measure, which is the unique measure invariant under the left and right action of $U(N)$ on itself. Such a probability space is often known as the Circular Unitary Ensemble (CUE). Let $\Lambda(z)$ be the characteristic polynomial of a matrix in the CUE. In recent years evidence has been accumulated suggesting that, in the limit as $T \to \infty$, the local statistical properties of $\zeta(s)$ can be modeled by the characteristic polynomials of matrices in the CUE where $N \approx \log(T/2\pi)$. The connection between the Riemann zeta-function and characteristic polynomials is extensive; examples include the distribution of the zeros of $\zeta(s)$, its value distribution and its moments. (For a series of review articles on the subject see [15] and references therein.)

Assuming that random matrix theory (RMT) provides an accurate description of $\zeta(s)$, the horizontal distribution of the zeros of $\zeta'(s)$ in proximity of the critical line should be the same as the radial distribution of the roots of $\Lambda'(z)$ close to the unit circle. This idea was first developed by Mezzadri [13], who determined the distribution of the zeros of $\Lambda'(z)$ that...
are very far from the unit circle and conjectured the leading order term of the distribution very close to the unit circle. In this paper we prove his conjecture. We also perform an analogous calculation for the Riemann zeta-function and conjecturally find that the result agrees with the RMT model. In addition, we do numerical computations in both cases and find a surprising feature in the distribution of zeros of the derivative, namely that the probability distribution is bimodal.

2. The zeros of $\zeta'(s)$.

We have mentioned that the main motivation for studying the zeros of the derivative of the Riemann zeta-function is its connection with RH. Indeed, Levinson and Montgomery’s result is the basis for Levinson’s method [10], which Conrey [2] used to prove that at least 40% of the zeros of the zeta-function are on the line $\sigma = \frac{1}{2}$.

Levinson’s method involves estimating a weighted average of the zeros of $\zeta'(s)$ to the left of $\frac{1}{2} + a/\log T$ for some fixed $a > 0$. Thus, zeros of $\zeta'(s)$ in the region $\frac{1}{2} \leq \sigma < \frac{1}{2} + a/\log T$ are an inherent loss in Levinson’s method. It would be useful to understand the magnitude of this loss. Alternatively, if we could find a lower bound for the number of zeros of $\zeta'$ in this region we could improve the estimate for the number of zeros on the critical line.

![Zeros of $\zeta(s)$ (dot), $\zeta'(s)$ (triangle), $\zeta''(s)$ (square), and $(\zeta'/\zeta)'(s)$ (star), with imaginary parts in the range $10^{15} < T < 10^{40}$.](image)

Figure 2.1 gives a representative example of the location of zeros of $\zeta'(s)$ and the relationship to the zeros of $\zeta(s)$ and various other derivatives of the zeta-function. It illustrates that zeros of $\zeta'(s)$ close to the critical-line correspond to closely spaced zeros of $\zeta(s)$. We make this statement precise in Section 6. Also, a zero of $\zeta'(s)$ seems to be “missing” when zeros of $\zeta$ are particularly far apart or when there are two successive large gaps. Indeed, there
can’t be a zero of $\zeta'(s)$ between every pair of zeros of $\zeta(s)$ because the density of zeros of $\zeta(s)$ is $\frac{1}{2\pi} \log(T/2\pi)$ while the density of zeros of $\zeta'(s)$ is $\frac{1}{2\pi} \log(T/4\pi)$. So on average there is a “missing” zero of $\zeta'(s)$ in each $T$ interval of width $2\pi/\log 2 \approx 9.06$.

Conrey and Ghosh [4] and subsequently Guo [7] improved Levinson and Montgomery’s result (1.1) and showed that a positive proportion of the zeros of $\zeta'(s)$ are much closer to the line $\sigma = \frac{1}{2}$. Indeed, for any fixed $a > 0$, the region

$$\sigma - \frac{1}{2} \geq \frac{a}{\log T}$$

contains a positive proportion of the zeros. Soundararajan [18] made further progress and introduced the functions

$$m^-(a) := \liminf_{T \to \infty} \frac{1}{N_1(T)} \sum_{0 < \gamma' \leq T} 1,$$

$$m^+(a) := \limsup_{T \to \infty} \frac{1}{N_1(T)} \sum_{0 < \gamma' \leq T} 1,$$

where $N_1(T)$ is the number of zeros $\beta' + i\gamma'$ of $\zeta'(s)$ with $0 < \gamma' \leq T$. Soundararajan proved that $m^-(a) > 0$ for $a > 2.6$, and conjectured that

$$m(a) = m^-(a) = m^+(a).$$

He also conjectured that $m(a)$ is continuous, that $m(a) > 0$ for all $a > 0$, and that $m(a) \to 1$ as $a \to \infty$. Zhang [23], Feng [6], Garaev and Yıldırım [8], and Ki [9] proved refinements of Soundararajan’s results. In particular Feng [6] showed that $m^-(a) > 0$ for all $a > 0$ unconditionally of RH but assuming a conjecture on the frequency of small gaps between consecutive critical zeros of $\zeta(s)$.

### 3. Zeros of derivatives of polynomials and statement of results

Suppose $f(z)$ is a polynomial with all zeros on the unit circle. (Eventually, $f$ will be a random polynomial obtained as the characteristic polynomial of a random unitary matrix.) The Gauss-Lucas theorem assures that all the roots of $f'(z)$ lie on or inside the unit circle (zeros of $f'$ on the circle occur only if $f$ has multiple zeros). If $f(z)$ has two zeros which are very close together, then $f'(z)$ will have a zero close by. (This is a consequence of the continuous dependence of the zeros of $f'$ on those of $f$. This dependence is actually piecewise analytic as will be described below.) The specific location of the nearby zero of $f'(z)$ will depend primarily on how close those two zeros of $f(z)$ are, and on the general position of the remaining zeros of $f(z)$. Thus, to leading order (with respect to the size of the gap), the distribution of zeros of $f'(z)$ near $|z| = 1$ should largely depend on the distribution of small gaps between zeros of $f(z)$, that is, on the tail of the nearest-neighbor spacing of zeros of $f(z)$. We will make this idea precise and treat in detail the case where $f(z) = \Lambda(z)$ is the characteristic polynomial of a CUE matrix (a matrix chosen from the unitary group $U(N)$, uniformly with respect to Haar measure).

Let $z'$ be a root of $\Lambda'(z)$ and define the random variable

$$S := N(1 - |z'|).$$
Denote by \( Q(s; N) \) the probability density function (p.d.f.) of \( S \). Mezzadri [13] showed that the limit
\[
Q(s) := \lim_{N \to \infty} Q(s; N) \tag{3.2}
\]
exists, and proved that
\[
Q(s; N) \sim \frac{1}{s^2}, \quad N \to \infty, \quad s \to \infty, \tag{3.3}
\]
with \( s = o(N) \). He also conjectured that
\[
Q(s) \sim \frac{4}{3\pi} s^{1/2}, \quad s \to 0. \tag{3.4}
\]

Formula (3.3) can be interpreted as the RMT counterpart of the Levinson-Montgomery bound (1.1) for the roots of \( \zeta'(s) \). The RMT model of the Riemann-zeta function is based on the observation that the local correlations of the non-trivial zeros of \( \zeta(s) \) coincide with those of the eigenvalues of matrices in the CUE. In order to make this correspondence quantitative, the densities of the eigenvalues and of the zeros of \( \zeta(s) \) must be made (asymptotically) equal, i.e.,
\[
\frac{N}{2\pi} = \frac{1}{2\pi} \log \frac{T}{2\pi}. \tag{3.5}
\]

It follows from (3.3) that the expected value of \( S \) does not exist—its p.d.f. does not decay sufficiently rapidly. On the other hand, using (3.3), the average of the values of \( S \) not exceeding \( N \) is
\[
\sim \int_1^N s \cdot \frac{ds}{s^2} = \log N, \quad N \to \infty. \tag{3.6}
\]

Recalling the relation (3.1) between \( S \) and \( |z'| \), we conclude that a positive proportion of the roots of \( \Lambda'(z) \) must lie within a distance from the unit circle bounded from above by
\[
(1 + \epsilon) \frac{\log N}{N} \tag{3.7}
\]
from the unit circle. Because of (3.5), formula (3.7) corresponds to Levinson and Montgomery’s result (1.1).

Now consider the roots \( e^{it_1}, \ldots, e^{it_N} \) (with \( -\pi < t_i \leq \pi \)) of the characteristic polynomial \( \Lambda(z) \) of a random unitary matrix distributed with Haar measure. It is convenient for our purposes to define
\[
x_j = \frac{N t_j}{2\pi}, \quad j = 1, \ldots, N, \tag{3.8}
\]
so that, on average, the distance between two consecutive \( x_j \)’s is one. The joint probability density function (j.p.d.f.) of the eigenvalues is given in terms of \( x_1, \ldots, x_N \) as
\[
p_2(x_1, \ldots, x_N) := \frac{1}{N^N N!} \prod_{1 \leq j < k \leq N} |e_N(x_k) - e_N(x_j)|^2, \tag{3.9}
\]
where we have used the notation \( e_N(x) := \exp(2\pi i x/N) \). Relabeling the indexes \( j = 1, \ldots, N \), if necessary, we assume that
\[
x_1 \leq \ldots \leq x_N < x_1 + N. \tag{3.10}
\]
We also extend the sequence \( \{x_j\} \) to be periodic by setting \( x_{N+1} = x_1, \ x_{N+2} = x_2, \) etc. Fix integers \( n \) and \( j \) with \( 0 \leq n \leq N - 2 \). Let us denote by \( p_2(n; s) \) the probability density function of \( x_{j+n+1} - x_j \). Since the j.p.d.f. (3.9) is invariant under translations, which means
\[
p_2(x_1 + \alpha, \ldots, x_N + \alpha) = p_2(x_1, \ldots, x_N) \quad \text{for all } \alpha \in \mathbb{R},
\]
it follows that \( p_2(n; s) \) does not depend on \( j \). If \( n = 0 \), \( p_2(s) := p_2(0; s) \) is known as the spacing distribution. It has an asymptotic expansion in powers of \( s \):
\[
p_2(p_2(s) = \left(1 - \frac{1}{3N^2}\right) \pi^2 s^2 - \left(\frac{2}{45} - \frac{1}{9N^2} + \frac{1}{15N^4}\right) \pi^4 s^4
\]
\[
+ \left(\frac{1}{315} - \frac{2}{135N^2} + \frac{1}{45N^4} - \frac{2}{189N^6}\right) \pi^6 s^6 + O(s^7).
\]
This expansion follows from the pair correlation function for \( U(N) \) (see [15]),
\[
\frac{\sin^2(\pi y)}{N^2 \sin^2(\pi y/N)}
\]
and the fact that the pair correlation function and the nearest neighbor spacing for \( U(N) \) agree to order 6.

To describe our main result, suppose that a root of \( \Lambda(z) \) is degenerate (which means \( x_{j+1} = x_j \)) so that \( z' = \exp(2\pi i x_j/N) \) will also be a root of \( \Lambda'(z) \). Simple considerations of continuity show that, if \( x_j \) and \( x_{j+1} \) are slightly moved apart (say, while keeping the remaining roots of \( f \) fixed), then \( z' \) will also move, but will still be close to the midpoint of the segment joining \( \exp(2\pi i x_j/N) \) to \( \exp(2\pi i x_{j+1}/N) \). In Proposition 8.1 we make this precise, showing that \( z' \) stays close to that midpoint provided \( x_{j+1} - x_j < 1/\pi \). Henceforth we assume that the rescaled distance
\[
\theta := x_{j+1} - x_j
\]
is small.

By the translation invariance of the j.p.d.f. of the \( x_j \)'s, we may assume without loss of generality that
\[
x_{j+1} = \frac{\theta}{2} \quad \text{and} \quad x_j = -\frac{\theta}{2}
\]
Define
\[
\delta := N(1 - z').
\]
In section 5 we shall show that
\[
\delta = b_1 \pi^2 \theta^2 + b_2 \pi^4 \theta^4 + O(\theta^6), \quad \text{as } \theta \to 0,
\]
where \( b_1 \) and \( b_2 \) are explicit functions of the zeros \( x_k \) for \( k \neq j, j + 1 \).

By combining the distribution of \( \theta \) given in (3.12) with information we will determine about \( b_1 \) and \( b_2 \) in Section 5, we will prove

**Theorem 3.1.** Let \( \Lambda(z) \) be the characteristic polynomial of a random matrix in \( U(N) \) distributed with respect to Haar measure. The distribution of \( \delta = N(1 - |z'|) \) arising from closely spaced zeros of \( \Lambda(z) \) is given by
\[
\frac{4}{3\pi} s^{1/2} - \frac{82}{45\pi} s^{3/2} + O\left(s^{5/2}\right),
\]
as \( N \to \infty \).
Note that Theorem 3.1 refers to the small values of $\delta$ that arise from closely spaced zeros of the polynomial. The theorem does not account for all small values of $\delta$. That distinction is often missed, because examples such as shown in Figure 2.1 give the mistaken impression that zeros of the derivative very close to the unit circle can only arise from closely spaced zeros of the polynomial. Farmer and Ki [5] give examples of families of polynomials whose (rescaled) zeros are bounded away from each other, but for which the density function of $\delta$ vanishes like $C \cdot s$ as $s \to 0$. They also argue that any larger density of zeros of $z'$ near the unit circle must arise from closely spaced zeros of the polynomial. Therefore we have the following corollary of Theorem 3.1, which proves Mezzadri’s conjecture about the distribution of $|z'|$.

**Corollary 3.2.** Let $\Lambda(z)$ be the characteristic polynomial of a random matrix in $U(N)$ distributed with respect to Haar measure, and let $Q(s; N)$ be the p.d.f. of $S = N(1 - |z'|)$ for $z'$ a root of $\Lambda'(z)$. Then

$$Q(s) = \lim_{N \to \infty} Q(s; N) = \frac{4}{3\pi} s^{1/2} + O(s).$$

(3.19)

Note that it remains an unsolved problem to show that $Q(s)$ is a proper probability distribution. That is, to show $\int_0^{\infty} Q(s)ds = 1$. This is the random matrix analogue of Soundararajan’s conjecture $m(a) \to 1$ as $a \to \infty$.

4. Comparison with data

We compare our formulas with numerical data.

We generated Haar-random matrices in $U(N)$ using the simple algorithm described in [14]. Figure 4.1 shows the empirical distribution of the rescaled zeros of $\Lambda'$ for various size matrices. Figure 4.2 shows the empirical cumulative distribution function $I_p(x) = \int_0^x Q(s, 40)ds$ for $U(40)$ and a comparison with the tail of the empirical cumulative distribution function with our results, showing good agreement.

We would like to know the underlying cause of the curious “second bump” in the distribution of zeros of derivatives. This seems to be a completely general phenomenon. In Figure 4.3 we show the analogous distributions for characteristic polynomials of matrices from COE(40) and for degree-40 polynomials whose roots are independently and uniformly distributed on the unit circle. Both cases show the “second bump”, although not quite at the same location. In Figure 6.1 we find a similar shape for the distribution of zeros of $\zeta'$. 

5. Proof of theorem 3.1

Suppose $f(z)$ is a degree-$N$ polynomial having all zeros on the unit circle, for which two zeros $z_1$, $z_2$ are very close together. Then the derivative $f'(z)$ will have a zero close to the midpoint $(z_1 + z_2)/2$. This follows because, if $z_1 = z_2$ (that is, if $f(z)$ has a multiple root at $z_1$), then $z_1$ is also a root of the derivative $f'(z)$, and the roots of $f'$ are continuous functions of the roots of $f$. By a rotation we can assume that

$$f(z) = F(z)(z - e^{-i\Theta/2})(z - e^{i\Theta/2}),$$

(5.1)

where $F(z)$ does not have any zeros $e^{it}$ with $-\Theta/2 \leq t \leq \Theta/2$. The root of $f'$ near 1 is the root near 1 of

$$\frac{f'}{f}(z) = \frac{F'}{F}(z) + \frac{1}{z - e^{-i\Theta/2}} + \frac{1}{z - e^{i\Theta/2}},$$

(5.2)

and we denote that root by $z' = 1 - \Delta$. 

We are concerned with the case when $f$ is the characteristic polynomial of a random CUE matrix, and we want to understand the distribution of the zeros of $f'$. Since the CUE measure (i.e., normalized Haar measure on $U(N)$) is invariant under rotation, the distribution depends only on the absolute value of the roots of $f'$. Those roots will accumulate near the unit circle as $N$ grows, so we must rescale them suitably in order to get a meaningful result in the limit $N \to \infty$. We let
\[ \Theta = 2\pi \theta/N, \]
\[ \Delta = \delta/N. \]  
\hspace{1cm} (5.3)
Note that this rescaling gives $\langle \theta \rangle = 1$. The rescaling of $\delta$ is more subtle, and it will be found in equation (5.12) that this is the correct rescaling. Note that, while $\theta$ is a real number, $\delta$ is usually complex (but typically with small imaginary part, as we shall see).

We will determine the leading order behavior of the roots which are close to the unit circle, so in the above notation we are interested in

$$ N(1 - |z'|) = N(1 - |1 - \Delta|) $$
$$ = \text{Re}(\delta) - \frac{\text{Im}(\delta)^2}{2N} + O\left(\frac{\delta^3}{N^2}\right). $$

(5.4)

5.1. Expansion for the roots. Exploiting the symmetry of the j.p.d.f. (3.9) under arbitrary relabellings of the variables $x_j$, as well as its translation invariance (cf., Section 3.2), we will assume (without loss of generality) that $F(z)$ from Equation (5.1) is given in the form (recall that $e_N(x) := \exp(2\pi i x/N)$)

$$ F(z) = \prod_{n=1}^{N-2} (z - e_N(x_n)), \quad x_n \in \left(-\frac{N}{2}, \frac{\theta}{2}\right] \cup \left[\frac{\theta}{2}, \frac{N}{2}\right), $$

(5.5)

(We are not excluding the possibility that $F$ has roots at $e_N(\pm\theta/2)$.) Therefore,

$$ \frac{F'(z)}{F(z)} = \sum_{n=1}^{N-2} \frac{1}{z - e_N(x_n)}. $$

Now let $z = z'$ be a root of $f'$ (say, that root which is closest to $z = 1$). The Implicit Function Theorem shows that $z'$ is an analytic function of $\theta$ (at least for $\theta$ sufficiently small). Indeed, fixing $F$ and regarding $f'$ as a function $f'(\theta; z)$ of both $\theta$ and $z$, we have $f'(0; 1) = 0$; by the assumption that $F$ has no root at $z = 1$ all that remains to observe is that

$$ 0 \neq 2F(1) = \frac{\partial}{\partial z} f'(0; 1) = f''(0; 1). $$

(5.6)

if $N$ is sufficiently large, by Proposition 8.1 $z'$ is defined uniquely and analytically as a function of $\theta$ in the domain

$$ |\theta| < \min\{\frac{1}{\pi}, |x_1|, \ldots, |x_{N-2}|\}. $$

(5.7)
We write \( z' = 1 - \delta/N \) and wish to expand \( (1 - \delta/N - e_N(x_n))^{-1} \) as a Taylor series in \( \delta \). This will be justified when \( |\delta| < N|1 - e_N(x_n)| \). Hence, we have

\[
\frac{F'}{F} \left( 1 - \frac{\delta}{N} \right) = N \sum_{j=0}^{\infty} A_j \delta^j
\]

for \( \delta \) sufficiently small, where

\[
A_j = (-1)^j \frac{1}{j!N^{j+1}} \left( \frac{F'}{F} \right)^{(j)} (1)
\]

\[
= \frac{1}{N^{j+1}} \sum_{n=1}^{N-2} \frac{1}{(1 - e_N(x_n))^{j+1}}.
\]

We will see later that the prefactor of \( N \) is the right choice to make the coefficients \( A_j \) approximately bounded. Note that, for \( |\theta| < 1 \) (equivalently, for \( |\Theta| < 2\pi/N \)), any \( \delta \) such that \( |\delta| < |\theta| = N|\Theta|/(2\pi) \) makes the expansion in (5.8) valid (independently of the exact location of the zeros of \( F \)), since

\[
|\delta| < \frac{N\Theta}{2\pi} \leq N|1 - e_N(\theta/2)| \leq N|1 - e_N(x_n)|, \quad 1 \leq n \leq N - 2.
\]

We also wish to expand the other terms in (5.2) as a series in \( \delta \). We have

\[
\frac{1}{z' - e^{-i\theta/2}} + \frac{1}{z' - e^{i\theta/2}} = \frac{2 - \frac{2\delta}{N} - 2 \cos \left( \frac{\pi \theta}{N} \right)}{2 - \frac{2\delta}{N} + \frac{4\delta^2}{N^2} - 2 \left( 1 - \frac{\delta}{N} \right) \cos \frac{\pi \theta}{N}}
\]

\[
= \frac{-2\delta + \pi^2\theta^2N^{-1} - \frac{1}{12}\theta^4\pi^4N^{-3} + O(\theta^6N^{-5})}{\delta^2 + \pi^2\theta^2 - \pi^2\delta\theta^2N^{-1} - \frac{1}{12}\theta^4\pi^4N^{-2} + \frac{1}{12}\pi^4\delta^4\theta^4N^{-3} + O(\theta^6N^{-4})}.
\]

Combining this with equations (5.2) and (5.8), putting all terms over a common denominator, and using the fact that \( f'(z') = 0 \), we have

\[
0 = \sum_{j=0}^{\infty} A_j \delta^j \left( \delta^2 + \pi^2\theta^2 - \pi^2\delta\theta^2N^{-1} - \frac{1}{12}\theta^4\pi^4N^{-2} + \frac{1}{12}\pi^4\delta^4\theta^4N^{-3} + O(\theta^6N^{-4}) \right)
\]

\[
- 2\delta + \pi^2\theta^2N^{-1} - \frac{1}{12}\pi^4\theta^4N^{-3} + O(\theta^6N^{-5}).
\]

Note that a global factor of \( N \) canceled to give the above equation, suggesting that we have chosen the correct scaling for \( \delta \).

Equation (5.12) is simply a more explicit and manageable form of the equation \( f'(\theta; z') = 0 \) defining \( z' \) implicitly as a function of \( \theta \). Noting that \( f'(0; 1) = 0 \), together with the functional equation \( f'(\theta; z) = f'(-\theta; z) \), it follows that \( \delta = \delta(\theta) \) has an expansion in powers of \( \theta^2 \), with no constant term, of the form

\[
\delta = b_1 \pi^2\theta^2 + b_2 \pi^4\theta^4 + O(\theta^6).
\]

From (5.12), we obtain

\[
0 = \left( A_0 - 2b_1 + \frac{1}{N} \right) \pi^2\theta^2
\]

\[
+ \left( A_1 b_1 + A_0 b_2^0 - 2b_2 - \frac{A_0 b_1}{N} - \frac{A_0}{12N^2} - \frac{1}{12N^3} \right) \pi^4\theta^4 + O(\theta^6).
\]
Setting each term in (5.14) equal to 0 and solving for $b_1$ and $b_2$ we have the following:

**Proposition 5.1.** In the notation above, if $0 \leq \theta < 1/\pi$ and $N$ is sufficiently large then $
abla = b_1 \pi^2 \theta^2 + b_2 \pi^4 \theta^4 + O(\theta^6)$ where

$$b_1 = \frac{A_0}{2} + \frac{1}{2N}$$

$$b_2 = \frac{1}{8} \left( A_0^2 + 2A_0 A_1 \right) + \frac{A_1}{4N} - \frac{A_0}{6N^2} - \frac{1}{24N^3},$$

with $A_j$ given in (5.9).

Note that in this analysis we have treated $F$ (and hence $A_j$) as being fixed, in the sense that we assume its zeros do not vary with $\theta$. In the next section we will show that when $f(z)$ is the characteristic polynomial of a matrix drawn from the CUE, this can be justified up to $O(\theta^7)$.

Using (5.13) and (5.15), we can determine the distribution of $\nabla$ from the distributions of $\theta$ and the $A_j$. For small $\theta$, this comes from the tail of the nearest-neighbor spacing.

5.2. **Nearest-neighbor spacing.** Proposition 5.1 provides a formula for $\nabla = N(1 - z')$, but what we really want is the distribution of $\delta^* := N(1 - |z'|)$. So by (5.4) and (5.13), and writing $B_j = \text{Re}(b_j)$, we have

$$\delta^* = B_1 \pi^2 \theta^2 + B_2 \pi^4 \theta^4 + O \left( \theta^6 + \frac{|\delta|^2}{N} + \frac{|\delta|^3}{N^2} \right)$$

$$= B_1 \pi^2 \theta^2 + B_2 \pi^4 \theta^4 + O \left( \theta^6 + \frac{\theta^2}{N} \right).$$

The second line is a corollary of Proposition 8.1, because $\delta \ll \theta$ if $\theta < 1/\pi$.

Suppose for the moment that $B_1$ and $B_2$ were constants (instead of being random). We would have $\delta^* = g(\theta) = B_1 \pi^2 \theta^2 + B_2 \pi^4 \theta^4 + O_N(\theta^6)$ where $\theta$ is random with p.d.f. (3.12) given by the nearest neighbor spacing of eigenvalues of unitary matrices. Then the distribution function of $\delta^*$ would be given by

$$\frac{p_2(g^{-}(s))}{g'(g^{-}(s))} = B_1^{-3/2} \frac{1}{6\pi} \left( 1 - \frac{1}{N^2} \right) s^{1/2}$$

$$- \frac{1}{\pi} \left( B_1^{-5/2} \left( 1 - \frac{5}{2N^2} + \frac{3}{2N^4} \right) + \frac{5B_1^{-7/2}B_2}{12} \left( 1 - \frac{1}{N^2} \right) \right) s^{3/2} + O_N(s^{5/2}).$$

It turns out that $B_1$ actually is a constant: in Section 5.3 we show that $B_1 = \frac{1}{4}$. It is fortunate that $B_1$ is a constant, otherwise it could be difficult to determine the expected value of quantities like $B_1^{-3/2}$.

The contribution of $B_2$ takes a bit more work. If $B_2$ was independent of $\theta$ then we could just average over the possible contributions of $B_2$ to (5.19). That is, in (5.19) replace $B_2$ by its expected value. This can be computed from the expected values of various combinations of $A_0$ and $A_1$. But $B_2$ is not independent of $\theta$. However, it is independent of $\theta$ to leading order. Our specific concern is the distribution of the other roots when $\theta$ is very small. This approximates the polynomial having a double zero. Since the next-nearest-neighbor spacing of $U(N)$ eigenvalues vanishes to order 7, the dependence of $B_2$ on $\theta$ is only to order
Thus, for the terms we are computing for the distribution of \( \delta^* \) we can treat \( B_2 \) as independent of \( \theta \). The expected value of \( B_2 \) is calculated in Section 5.3.

5.3. Expected value of \( A_j \). We have

\[
A_j := \frac{1}{N^{j+1}j!} \left( \frac{F^*}{F} \right)^{(j)} (1)
\]

\[
= (-1)^j N^{-j-1} \sum_{n=1}^{N-2} \frac{1}{(1 - e^{it_n})^{j+1}},
\]

where \( t_1, t_2, \ldots \) are the arguments of the zeros of \( F(z) \). Since

\[
\frac{1}{1 - e^{it}} = \frac{1}{2} + i \frac{1}{2} \cot \left( \frac{t}{2} \right)
\]

we see that

\[
\text{Re}(A_0) = \frac{N - 2}{2N},
\]

so

\[
B_1 = \text{Re}(b_1) = \frac{1}{4},
\]

as claimed. Note also that

\[
\langle A_0 \rangle = \frac{1}{2} - \frac{1}{N},
\]

because the imaginary part of the summand is odd.

For \( A_j \) with \( j \geq 1 \), we require a random matrix calculation. The sum in (5.20) is dominated by the terms where \( e^{it_n} \) is close to 1, so one possibility is to determine the level densities of the \( t_n \). We will find the expected value of \( A_j \) by appealing to prior results on averages of ratios of characteristic polynomials [3].

We assume that \( f(z) \) is the characteristic polynomial of a matrix chosen uniformly with respect to Haar measure from the unitary group \( U(N) \). We restrict to those matrices which have two eigenvalues very close to 1, and we wish to determine the joint distribution of the remaining eigenvalues. This is very similar to the calculations of Dueñez [16] and Snaith [17] for the orthogonal group \( SO(N) \).

First we restrict the measure on the entire ensemble to determine the measure on the remaining eigenvalues. Haar measure on \( U(N) \) is given by

\[
d\mu = C \prod_{1 \leq n < m \leq N} |e^{it_n} - e^{it_m}|^2 dt_1 \cdots dt_N.
\]

Here and following, \( C \) is a normalization constant which may vary from line to line, chosen so that the measure has total mass 1. Restricting to those matrices which have one eigenvalue at 1 is equivalent to rotating (changing variables) to move an eigenvalue to 1. So we can also write the measure as

\[
d\mu_1 = C \prod_{1 \leq n < m \leq N-1} |e^{it_n} - e^{it_m}|^2 \prod_{1 \leq n \leq N-1} |e^{it_n} - 1|^2 dt_1 \cdots dt_{N-1}.
\]

The set of matrices which have a repeated eigenvalue at 1 has measure zero, so there is no canonical way to restrict the measure. However, we are interested in the limiting case of two
eigenvalues which are very close together, so we determine the measure by restricting the measure (5.26) to have 
\[ |t_{N-1}| \leq t_{N-1} \leq |t| \], and then let \( t \to 0 \). The resulting measure is

\[
d\mu_2 = C \prod_{1 \leq n < m \leq N-2} |e^{it_n} - e^{it_m}|^2 \prod_{1 \leq n \leq N-2} |e^{it_n} - 1|^4 \, dt_1 \cdots dt_{N-2}. \tag{5.27}
\]

Let \( U_2(N-2) \) denote the ensemble of unitary matrices with joint eigenvalue measure \( \mu_2 \). Then if \( g = g(e^{it_1}, \ldots, e^{it_{n-2}}) \) we have

\[
\langle g \rangle_{U_2(N-2)} = C_N \langle |\Lambda(1)|^4 \rangle_{U(N-2)} \tag{5.28}
\]

where the right side is a Haar measure average, \( \Lambda \) is the characteristic polynomial, and

\[
C_N = \langle |\Lambda(1)|^4 \rangle_{U(N-2)}^{-1}. \tag{5.29}
\]

In other words, an expectation involving repeated eigenvalues on \( U(N) \) is equivalent to an expectation on \( U(N-2) \) with an extra factor of the 4th power of the characteristic polynomial. This is the key observation for computing the expected values of the \( A_j \) because it reduces it to the evaluation of known quantities.

Specifically, let

\[
G(\alpha_1, \alpha_2, \alpha_3, \alpha_4; \beta_1, \beta_2; \gamma_1, \gamma_2) = \frac{\Lambda(e^{-\alpha_1})\Lambda(e^{-\alpha_2})\Lambda(e^{-\alpha_3})\Lambda(e^{-\alpha_4})\Lambda(e^{-\beta_1})\Lambda(e^{-\beta_2})}{\Lambda(e^{-\gamma_1})\Lambda(e^{-\gamma_2})}. \tag{5.30}
\]

Theorem 4.1 of [3] provides an explicit formula for the expected value of \( G \) for \( \Lambda \) the characteristic polynomial of Haar distributed matrices on \( U(N-2) \). The formula is complicated so we do not reproduce it here. This is sufficient to determine the expected values of all the quantities in (5.15). The calculation requires the assistance of a computer algebra package.

We now present the answers, which we determined with the help of Mathematica.

The normalization constant for the measure \( \mu_2 \) is (the reciprocal of)

\[
\langle G(0, 0, 0, 0; 0, 0, 0, 0) \rangle_{U(N-2)} = \langle |\Lambda(1)|^4 \rangle_{U(N-2)} = \frac{N^4}{12} - \frac{N^2}{12} = C_N^{-1}, \tag{5.31}
\]
say. As \( \theta \to 0 \) we have

\[
\langle A_0^3 \rangle = \left\langle \left( \frac{\Lambda'}{\Lambda} \right)^3 (1) \right\rangle_{U_2(N-2)} = -C_N \frac{\partial^3}{\partial \alpha_1 \partial \alpha_2 \partial \gamma_1} \left. \right|_{(\alpha_1, \alpha_2, \gamma_1) = (0, 0, 0)} \langle G(\alpha_1, \alpha_2, 0; 0, 0; \gamma_1, 0) \rangle_{U(N-2)} = \frac{1}{10} N^3 - \frac{7}{10} N^2 + \frac{8}{5} N - \frac{6}{5}.
\]

\[
\langle A_1 \rangle = \left\langle \left( \frac{\Lambda'}{\Lambda} \right)' (1) \right\rangle_{U_2(N-2)} = C_N \left( 1 + \frac{d}{d \alpha_1} \right) \left. \right|_{\alpha_1 = 0} \frac{\partial}{\partial \alpha_1} \left. \right|_{\gamma_1 = \alpha_1} \langle G(\alpha_1, 0, 0, 0; \gamma_1, 0) \rangle_{U(N-2)} = \frac{1}{15} N^2 - \frac{1}{2} N + \frac{11}{15}.
\]

\[
\langle A_0 A_1 \rangle = \left\langle \left( \frac{\Lambda'}{\Lambda} \right) (1) \left( \frac{\Lambda'}{\Lambda} \right)' (1) \right\rangle_{U_2(N-2)} = -C_N \left( 1 + \frac{d}{d \alpha_1} \right) \left. \right|_{\alpha_1 = 0} \frac{\partial}{\partial \alpha_1} \left. \right|_{\gamma_1 = \alpha_1} \frac{\partial}{\partial \alpha_2} \left. \right|_{(\alpha_2, \gamma_2) = (0, 0)} \langle G(\alpha_1, \alpha_2, 0; 0, 0; \gamma_1, \gamma_2) \rangle_{U(N-2)} = \frac{1}{30} N^3 - \frac{3}{10} N^2 + \frac{13}{15} N - \frac{4}{5}.
\] (5.32)

Thus,

\[
\langle B_2 \rangle = \text{Re} \langle b_2 \rangle = \frac{1}{48} - \frac{7}{48} N^{-1} + O(N^{-2}).
\] (5.33)

Inserting this into (5.19) gives the expansion claimed in Theorem 3.1.

6. The Riemann zeta-function

We do analogous calculations for derivatives of the Riemann zeta-function and compare our results with data.

We start with

\[
\frac{\zeta'(s)}{\zeta(s)} = b - \frac{1}{s - 1} - \frac{1}{2} \frac{\Gamma'(\frac{1}{2}s + 1)}{\Gamma(\frac{1}{2}s + 1)} + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right)
\] (6.1)

where \( b = \log 2\pi - 1 - \frac{1}{2} \gamma \). As in the polynomial case, we assume there are two very closely spaced zeros of the \( \zeta \)-function and look for the nearby zero of \( \zeta' \). Suppose the closely spaced zeros are

\[
\rho_\pm = \frac{1}{2} + i(t \pm \frac{1}{2} \Theta)
\] (6.2)

with

\[
s' = \frac{1}{2} + X + it
\] (6.3)
a zero of $\zeta'$.

Using

$$\frac{\Gamma'}{\Gamma}(s) = \log(s) + O(1/s)$$

we have

$$0 = b^* - \frac{1}{2} \log t + \frac{1}{s' - \rho_-} + \frac{1}{s' - \rho_+} + \sum_{\rho \neq \rho\pm} \left( \frac{1}{s' - \rho} + \frac{1}{\rho} \right),$$

where

$$b^* = b + \frac{1}{2} \log(2) - i \frac{\pi}{4} + O(1/t).$$

Note that

$$1 \frac{1}{s' - \rho_-} + 1 \frac{1}{s' - \rho_+} = \frac{8X}{4X^2 + \Theta^2}$$

and

$$1 \frac{1}{s' - \rho} + 1 \frac{1}{\rho} = \frac{X - i(t - \gamma)}{X^2 + (t - \gamma)^2} + \frac{1}{4} \frac{1}{\gamma^2},$$

which has real part

$$\frac{X}{X^2 + (t - \gamma)^2} + \frac{1}{4} \frac{1}{\gamma^2}.\quad (6.9)$$

As in the polynomial case, we rescale:

$$X = x / \log t$$

$$\Theta = 2\pi \theta / \log t.$$  

(6.10)

Note that this is analogous to the rescaling in the unitary case because $N \approx \log(t/2\pi)$. We have

$$0 = b^{**} + I - \frac{1}{2} \log t + \frac{2x \log t}{x^2 + \pi^2 \theta^2} + x \log t \sum_{\gamma} \frac{1}{x^2 + 4\pi^2 \gamma^2},$$

where $b^{**}$ is a real constant, $I$ is purely imaginary, and the sum is over the rescaled zeros $\gamma = \log(t(t - \gamma))/2\pi$. We follow the same procedure as in the $U(N)$ case. First multiply through (6.11) by $(x^2 + \pi^2 \theta^2)/\log t$ and expand the final summand as a series in $x$, giving

$$0 = 2x + (x^2 + \pi^2 \theta^2) \left( -\frac{1}{2} + x\alpha_1 - x^3\alpha_2 + O(x^5) \right) + \text{smaller terms},$$

where

$$\alpha_j = \sum_{\gamma} \frac{1}{(4\pi^2 \gamma^2)^j}.$$  

(6.13)

Note that (6.12) has the same form as (5.12). Now write $x = \beta_1 \pi^2 \theta^2 + \beta_2 \pi^4 \theta^4 + O(\theta^6)$ and gather terms to get

$$\left(2\beta_1 - \frac{1}{2}\right) \pi^2 \theta^2 + \left(2\beta_2 - \frac{1}{2}\beta_1^2 + \beta_1 \alpha \right) \pi^4 \theta^4 + \text{smaller terms}.$$  

(6.14)

Thus,

$$\beta_1 \sim \frac{1}{4}$$

$$\beta_2 \sim \frac{1}{64} - \frac{1}{8} \alpha_1,$$  

(6.15)

(6.16)
which exactly corresponds to the $U(N)$ case in Proposition 5.1.

The sum over zeros $\alpha_1$ is similar to the expression for $A_1$ in the unitary case. We can determine the expected value of the sum if we assume CUE statistics for the zeros of the Riemann zeta-function, restrict to having two closely spaced zeros, and find the one-level density of the remaining zeros. That calculation is in Section 7. In the notation of Lemma 7.1 we have

$$\langle \alpha_1 \rangle = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{1}{t^2} W_1^{(2,0)}(t) \, dt = \frac{1}{15}. \quad (6.17)$$

Note that this is the same as the expected value of $A_1$.

Thus, assuming that the spacing of zeros of the Riemann zeta function has the same distribution as the spacing of eigenvalues of random unitary matrices, we find that the leading order behavior of zeros of $\zeta'$ near the $\frac{1}{2}$-line is the same as that of zeros of $\Lambda'$ near the unit circle.

In Figure 6.1 we show the empirical distribution of the zeros of $\zeta'$ for $10^6 < t < 10^6 + 60000$. The general shape of the distribution shows a striking similarity with the zeros of the derivative of characteristic polynomials of unitary matrices.

![Figure 6.1. Normalized distribution of the real part of the zeros of $\zeta'(s)$. Data is for the approximately 100000 zeros with imaginary part in $[10^6, 10^6 + 60000]$.](image)

7. Calculation of the 1-level density

We prove the following

**Lemma 7.1.** Fix $a, b > -1/2$. If $t_1, t_2, \ldots, t_M$ are independently distributed with respect to the probability measure

$$d\mu = d\mu^{(a,b)} = \frac{1}{C_{M,a,b}} \prod_{1 \leq n < m \leq M} |e^{it_n} - e^{it_m}|^2 \prod_{1 \leq n \leq M} |e^{it_n} - 1|^{2a} |e^{it_n} + 1|^{2b} dt_1 \cdots dt_M, \quad (7.1)$$

then the large-$M$ limiting (rescaled) 1-level density of the normalized values $\tilde{t}_j = t_j M/2\pi$ is given by

$$W_1^{(a,b)}(t) = \frac{\pi^2}{2} \left( J_{a-\frac{1}{2}}(\pi t)^2 + J_{a+\frac{1}{2}}(\pi t)^2 \right) - a\pi J_{a-\frac{1}{2}}(\pi t) J_{a+\frac{1}{2}}(\pi t). \quad (7.2)$$
The measure $d\mu^{(a,b)}$ is Haar measure on $U(M + a + b)$ restricted to those matrices which have $a$ eigenvalues equal to 1 and $b$ eigenvalues equal to $-1$. Note that $W_1^{(a,b)}$ is independent of $b$.

**Proof.** For fixed $a, b > -\frac{1}{2}$ define $\omega(z) = |z - 1|^{2a} |z + 1|^{2b}$. Let $\{\phi_n\}_{n=0}^\infty$ be the sequence in $\mathbb{C}[x]$ uniquely determined by the following requirements:

1. For all $n \geq 0$, $\phi_n$ is of degree $n$ and has positive leading coefficient.
2. For all $m, n \geq 0$, 
   \[
   \langle \phi_m, \phi_n \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_m(e^{it})\overline{\phi_n(e^{it})}w(e^{it})dt = \delta_{mn},
   \]
   where $\delta_{mn}$ is the Kronecker delta.

Then $\{\phi_n\}$ is the sequence of normalized orthogonal polynomials on the unit circle with respect to the measure $d\nu(z) = (2\pi)^{-1} \omega(z) d\ell(z)$ (where $d\ell$ is the arc-length element.)

Let

\[
K_M(z, w) := \sum_{n=0}^{M-1} \phi_n(z)\overline{\phi_n(w)}
\]

be the projection kernel onto polynomials of degree less than $M$ with respect to the inner product (7.3). By the Gaudin-Mehta method, the probability measure (7.1), when regarded as a measure on the unit circle, can be rewritten as

\[
d\mu_k = \frac{1}{M!} \det_{1 \leq j,k \leq M} (K_M(z_j, z_k)) \prod_{j=1}^{M} d\nu(z_j)
\]

(note that the normalization constant $C_{M,k}$ is no longer needed). Then the 1-level measure is $W_1^{(M)}(z) d\nu(z)$, where

\[
W_1^{(M)}(z) = K_M(z, z).
\]

(The normalization above is such that the total mass of the 1-level measure is equal to $M$.)

Let the “dual” $\phi^*$ of a polynomial $\phi(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$ of degree $n$ be the polynomial

\[
\phi^*(z) = z^n \overline{\phi(z^{-1})} = \overline{c_n} + \overline{c}_{n-1} + \cdots + \overline{c}_1 z^{n-1} + \overline{c}_0 z^n.
\]

Then we have the following formula of Szegő for the projection kernel ([21], Theorem 11.4.2):

\[
K_M(z, w) = \frac{\phi_M^*(\overline{z})\phi_M^*(w) - \overline{\phi_M(z)}\phi_M(w)}{1 - \overline{z}w}.
\]

(This formula is analogous to the classical one of Christoffel and Darboux for the projection kernel of orthogonal polynomials on the line.)

In view of (7.6) and (7.8), in order to find the rescaled limit of the 1-level measure as $M \to \infty$ it will suffice to derive the asymptotic behavior of the orthogonal polynomials $\phi_n$ as $n \to \infty$ near the point $z = +1$. Theorem 7.2 and formula (7.16) below are the key ingredients, but first we need to introduce some notation.

Denote by $P_n^{(a,b)}$ the classical Jacobi polynomials: they are orthogonal in the interval $[-1, 1]$ with respect to the measure

\[
w^{(a,b)}(x) := (1 - x)^a (1 + x)^b
\]
and are normalized as follows ([21], Equation 4.3.3):
\[ h^{(a,b)}_n := \int_{-1}^{1} |P_n^{(a,b)}(x)|^2 w^{(a,b)}(x) \, dx = \frac{2^{a+b+1}}{2n + a + b + 1} \frac{\Gamma(n + a + 1)\Gamma(n + b + 1)}{\Gamma(n + a + b + 1)}. \] (7.10)

Let also
\[ h^+_n = 2^{a+b} h^{(a+1/2,b+1/2)}_n, \quad h^-_n = 2^{a+b} h^{(a-1/2,b-1/2)}_n. \] (7.11)

**Theorem 7.2.** [21]
\[
\begin{align*}
z^{-n} \phi_{2n}(z) &= AP_n^{(a-1/2,b-1/2)} \left( \frac{z + z^{-1}}{2} \right) + B(z - z^{-1})P_n^{(a+1/2,b+1/2)} \left( \frac{z + z^{-1}}{2} \right) \\
z^{-n+1} \phi_{2n-1}(z) &= CP_n^{(a-1/2,b-1/2)} \left( \frac{z + z^{-1}}{2} \right) + D(z - z^{-1})P_n^{(a+1/2,b+1/2)} \left( \frac{z + z^{-1}}{2} \right).
\end{align*}
\] (7.12)

Letting \( c_n = (a + b)/(n + a + b) \), we have
\[
\begin{align*}
A &= \sqrt{\frac{\pi}{2} \frac{1 + c_n}{h^-_n}} \\
B &= \frac{1}{2} \sqrt{\frac{\pi}{2} \frac{1 - c_n}{h^+_n}} \\
C &= \sqrt{\frac{\pi}{2} \frac{1 - c_n}{h^-_n}} \\
D &= \frac{1}{2} \sqrt{\frac{\pi}{2} \frac{1 + c_n}{h^+_n}}.
\end{align*}
\]

Equation (7.12) appears as (11.5.4) in Szegő’s book (except for an obvious typographical mistake therein.) The constants \( A, B, C, D \) can be easily found using Szegő’s equation (11.5.2) together with the fact that \( \phi_{2n-1} \) is a polynomial, which forces the coefficient of \( z^{-n} \) on the right-hand side of equation (7.12) to vanish. We omit the details.

A meaningful rescaling of the 1-level measure is achieved via the change of variables
\[ t = \frac{2\pi \xi}{M}. \] (7.13)

Indeed, one finds that the limiting 1-level measure \((as M \to \infty)\) is \( W_1(\xi)d\xi \), where
\[ W_1(\xi) = K_\infty(\xi, \xi), \] (7.14)
\[ K_\infty(\xi, \eta) = \lim_{M \to \infty} \frac{1}{M} K_M(e^{2\pi \xi/M}, e^{2\pi \eta/M})\sqrt{\omega(e^{2\pi \xi/M})\omega(e^{2\pi \eta/M})}. \] (7.15)

(Actually, all limiting local correlations can be expressed in terms of the limiting kernel \( K_\infty \), not just the 1-level density.)

It remains to compute \( K_\infty(\xi, \eta) \). It suffices to use equation (7.12) in formula (7.8) and an asymptotic formula by Szegő’s (see [21], equation (8.21.17)):
\[
\left( \sin \frac{t}{2} \right)^a \left( \cos \frac{t}{2} \right)^b P_n^{(a,b)}(\cos t) = N^{-a} \frac{\Gamma(n + a + 1)}{n!} \sqrt{\frac{t}{\sin t}} J_a(Nt) + t^{a+2}O(n^a),
\] (7.16)
valid for fixed \( a > -1, b \in \mathbb{R} \), in the range \( 0 < t \leq c/n \) for any fixed constant \( c > 0 \), where \( N = n + (a + b + 1)/2 \) and \( J_a \) is a Bessel function of the first kind.

A tedious but straightforward computation finally gives:
\[
\begin{align*}
K_\infty(\xi, \eta) &= \frac{\pi}{2} e^{i\pi(\eta - \xi)} \sqrt{\frac{\xi \eta}{\xi - \eta}} \left( J_{a+1/2}(\pi \xi)J_{a-1/2}(\pi \eta) - J_{a-1/2}(\pi \xi)J_{a+1/2}(\pi \eta) \right) \\
K_\infty(\xi, \xi) &= \frac{\pi}{2} \left\{ \pi [J_{a+1/2}(\pi \xi)^2 + J_{a-1/2}(\pi \xi)^2] - 2a J_{a+1/2}(\pi \xi)J_{a-1/2}(\pi \xi) \right\}.
\end{align*}
\] (7.17)
Remark. The kernel $K_{\infty}(\xi, \eta)$ can be used to compute $n$-level correlations and spacing statistics through (matrix or operator) determinants (see [22] for an explanation and proof of these applications). Incidentally, for the purposes of evaluating such determinants, the factor $e^{i\pi(n-\ell)}$ may be suppressed in (7.17) (this is tantamount to conjugating the corresponding integral operator by a unitary transformation). \hfill \Box

8. Appendix: The Domain of Analyticity of the Root of the Derivative

by Eduardo Dueñez

Define as before $e(x) := e^{2\pi ix}$ for $x$ real. For any fixed $N \geq 3$ define $e_N(x) := e(x/N) = e^{2\pi ix/N}$. Consider $N$ unit complex numbers

$$e_N(\theta_0), e_N(-\theta_0), e_N(\theta_1), e_N(\theta_2), \ldots, e_N(\theta_{N-2}),$$

where

$$0 \leq \theta_0 \leq \theta_j \leq N - \theta_0, \quad j = 1, 2, \ldots, N - 2.$$ (I. e., the open arc centered at 1 of the unit circle $|z| = 1$ having endpoints $e_N(\pm \theta_0)$ contains none of the remaining $N - 2$ numbers). The inequalities (8.2) define a "pyramid" $P_N$ contained in the cube $[0, N]^{N-1}$. For fixed $0 < T \leq N/2$, denote by $P_N^{(T)}$ the closed truncation of $P_N$ at height $T$. Then $P_N^{(T)}$ is defined by the inequalities

$$\theta_0 \leq \theta_j \leq N - \theta_0, \quad j = 1, 2, \ldots, N - 2;$$

$$0 \leq \theta_0 \leq T.$$ (8.3)

For notational convenience we will set $\theta_0 = (\theta_1, \theta_2, \ldots, \theta_{N-2})$ and $\theta_0 = (\theta_0; \Theta)$. Finally, let

$$f(z) = f(\Theta_0; z) = (z - e_N(\theta_0))(z - e_N(-\theta_0)) \prod_{j=1}^{N-2} (z - e_N(\theta_j))$$

be the monic polynomial of degree $N$ with roots (8.1).

In this section we prove the following result.

**Proposition 8.1.** With the above notation, for every $T < \frac{1}{\pi}$ there exists $N(T)$ such that for all $N \geq N(T)$ and for each $\Theta_0$ in $P_N^{(T)}$, the derivative $f'(z) = \frac{d}{dz}f(\Theta_0; z)$ of $f$ has a unique root $z' = \zeta(\Theta_0)$ in the open disk with diameter $[e_N(-\theta_0), e_N(\theta_0)]$. Moreover, $\zeta$ is an analytic function of $\Theta_0$ in the interior of $P_N^{(T)}$.

We remark that, by the Gauss-Lucas theorem, the root $z'$ alluded to in Proposition 8.1 must lie on or to the left of the vertical diameter $[e_N(-\theta_0), e_N(\theta_0)]$.

**Proof.** Fix $T < 1/\pi$ and consider $N$ as a parameter ($N \geq 3$) for the time being. Let

$$F(z) = F(\Theta; z) := \prod_{j=1}^{N-2} (z - e_N(\theta_j)),$$ (8.5)

so

$$f(z) = (z - e_N(\theta_0))(z - e_N(-\theta_0))F(z)$$ (8.6)

and

$$\frac{f'}{f}(z) = g(z) + L(\Theta; z),$$ (8.7)
where
\[ g(z) := \frac{1}{z - e_N(\theta_0)} + \frac{1}{z - e_N(-\theta_0)}, \]  
(8.8)
\[ L(\Theta; z) := \frac{F' - D}{F} (z) = \sum_{j=1}^{N-2} \frac{1}{z - e_N(\theta_j)}. \]  
(8.9)

Define \( c_N(x) := \cos(2\pi x/N) \) and \( s_N(x) := \sin(2\pi x/N) \), so \( e_N(x) = c_N(x) + is_N(x) \). Parametrize the boundary of the disk with diameter \( e_N(\pm \theta_0) \) (i.e., the disk \( |z - c_N(\theta_0)| \leq s_N(\theta_0) \)) as:
\[ z(\phi) = c_N(\theta_0) + i e^{i\phi} s_N(\theta_0). \]  
(8.10)

Since roots of \( f \) occur at \( e_N(\pm \theta_0) \) whenever \( f \) has multiple roots there, it is best to work instead with a slightly deformed contour \( C \) obtained as the boundary of the “twice bitten” disk
\[
\begin{align*}
|z - c_N(\theta_0)| &\leq s_N(\theta_0) \\
|z - e_N(\theta_0)| &\geq \epsilon \\
|z - e_N(-\theta_0)| &\geq \epsilon.
\end{align*}
\]

missing tiny \( \epsilon \)-neighborhoods of the points \( e_N(\pm \theta_0) \). We still assume that \( C \) is parametrized by (8.10), except for \( \phi \) in a small \( \delta \)-neighborhood of any multiple of \( \pi \). (We will not write down the exact parametrization of \( C \) for such values of \( \phi \) since its precise form will not be needed.)

Write \( C \) as a union of four pieces: \( C = C_d \cup C_r \cup C_u \cup C_\ell \) (down, right, up, left), where
\[
\begin{align*}
C_d &= \{ z(\phi) : \pi - \delta \leq \phi \leq \pi + \delta \}, \\
C_r &= \{ z(\phi) : -\pi + \delta \leq \phi \leq -\delta \}, \\
C_u &= \{ z(\phi) : -\delta \leq \phi \leq \delta \}, \\
C_\ell &= \{ z(\phi) : \delta \leq \phi \leq \pi - \delta \}.
\end{align*}
\]

The assumed inequalities (8.3) ensure that there are no zeros of \( f \) anywhere on \( C \) (and, a fortiori, no multiple zeros); hence, every zero of \( f' \) inside \( C \) is a zero of \( f'/f \) with the same multiplicity. It suffices to ensure that that (for all sufficiently small \( \epsilon \)) the contour \( C \) encloses exactly one zero of \( f'/f \). Note that, since \( f(\Theta; z) \) has no zeros on or inside \( C \), \( f'/f \) has no poles there either. By the Argument Principle, the claim in Proposition 8.1 regarding the uniqueness of the zero \( z' \) of \( f' \) is equivalent to showing that the image \( D \) of \( C \) under \( f'/f \) has index 1 about the origin for large \( N \) and all sufficiently small \( \epsilon \).

We let \( w(\phi) := \frac{L'}{L}(z(\phi)) \) and denote the images of the four pieces \( C_d, C_r, C_u, C_\ell \) of \( C \) under \( f'/f \) by \( D_d, D_r, D_u, D_\ell \).

The idea of the proof is very simple. As we shall show, the dominant part of the logarithmic derivative \( f'/f(z) \) (for \( z \) in \( C \)) is \( g(z) \). The image of \( C \) under \( g(z) \) is easy to describe explicitly; indeed, it is a curve \( \mathcal{E} \) having index 1 about the origin. The curve \( D \) can be regarded as a perturbation of \( \mathcal{E} \). We show that, if \( H \) and \( \epsilon \) are sufficiently small, then the remaining terms \( 1/(z - e_N(\theta_j)) \), \( j = 1, 2, \ldots, N - 2 \), are small enough to keep the index of \( D \) equal to that of \( \mathcal{E} \).

We denote by \( \mathcal{E}_d, \mathcal{E}_r, \mathcal{E}_u, \mathcal{E}_\ell \) the images under \( g \) of the respective parts of \( C \) (figure 8.1).
• $\mathcal{E}_r$ is parametrized as

$$g(z(\phi)) = \frac{1}{2s_N(\theta_0) \sin \phi}, \quad -\pi + \delta \leq \phi \leq -\delta. \quad (8.11)$$

Thus, $\mathcal{E}_r$ is a twice-traversed straight segment starting at the faraway point $H = g(z(-\pi+\delta)) = 1/(2s(\theta_0) \sin \delta)$, moving leftward to the point $A = g(z(0)) = 1/(2s(\theta_0))$ and retracing itself back to $B = g(z(-\delta))$ (here $B = H$).

• $\mathcal{E}_u$ is a large arc $BCD$ on the upper half-plane. ($\mathcal{E}_u$ is essentially a semicircle, for $\epsilon$ small. This is easily seen from the fact that the dominant term in $f'/f(z)$ for $z$ very close to $e_N(\theta_0)$ is $1/(z - e_N(\theta_0))$, namely an inversion with center $e_N(\theta_0)$, taking the tiny (almost) semicircle $\mathcal{C}_u$ to a huge (almost) semicircle $\mathcal{E}_u$. $\mathcal{E}_u$ escapes any bounded region of the plane as $\epsilon$ approaches zero.

• $\mathcal{E}_l$ and $\mathcal{E}_d$ are obtained from $\mathcal{E}_r$ and $\mathcal{E}_u$ through central symmetry with respect to the origin.

It is clear that $\mathcal{E}$ has index 1 about the origin for all sufficiently small $\epsilon$.

For notational convenience, denote by $\tilde{A} = w(-\pi/2), \tilde{B} = w(-\delta), \ldots, \tilde{H} = w(-\pi+\delta)$ the points on $\mathcal{D}$ analogous to $A, B, \ldots, H$ on $\mathcal{E}$.

We claim that if $N$ is large enough and $\Theta_0 \in \mathcal{P}_N^{(T)}$, then the index of $\mathcal{D}$ about the origin is also 1 for all sufficiently small $\epsilon$ and all $N \geq N(T)$. Note that $\epsilon$ is allowed to depend on $\Theta_0$, whereas the lower bound $N(T)$ for $N$ depends only on $T$.

For notational simplicity, we set $\theta_{N-1} := -\theta_0$. Write

$$\frac{f'}{f}(z) = \frac{m}{z - e_N(\theta_0)} + \sum_{j=m}^{N-1} \frac{1}{z - e_N(\theta_j)},$$

where $m \geq 1$ is the multiplicity of the zero $e_N(\theta_0)$ of $f(z)$ and $\theta_j, j = m, \ldots, N-1$, are the remaining zeros of $f(z)$.

Let $\epsilon_0 > 0$ be the minimum of the distances from $e_N(\theta_0)$ to the other $e_N(\theta_j), m \leq j \leq N - 1$. Now let $\epsilon = \epsilon_0/N$. If $|z - e_N(\theta_0)| = \epsilon$, then $|z - e_N(\theta_j)| \geq \epsilon_0$ ($m \leq j \leq$
By Lemma 8.2, \(|R(\theta_0; z)| \leq (N - m)/\epsilon_0\). Moreover, \(m/(z - e_N(\theta_0))\) maps \(C_u\) into (part of) the upper half-circle \(S\): \(|w| = m/\epsilon, \text{Re} w > 0\). Therefore, \(D_u\) will be supported on the \(N-1\)-neighborhood \(T\) of \(S\). Since \(T\) intersects the imaginary axis \(\text{Re} w = 0\) on the interval \(i \left[ \frac{m}{\epsilon} - \frac{N-m}{\epsilon_0}, \frac{m}{\epsilon} + \frac{N-m}{\epsilon_0} \right] = i \left[ \frac{m+N(m-1)}{\epsilon_0}, \frac{N+m(N-1)}{\epsilon_0} \right]\), our claim that \(D_u\) only intersects the positive imaginary axis is proved. The claim that \(D_\ell\) only intersects the negative imaginary axis for suitably small \(\epsilon\) is proved along identical lines.

In order to prove the claim that \(D\) has index 1 about the origin, it suffices now to show that \(D_\ell\) is contained in the right half-plane \(\text{Re}(w) > 0\) and \(D_\ell\) in the left half-plane \(\text{Re}(w) < 0\). The truth of said statement for \(D_\ell\) is obvious, since each of the terms \(1/(z - e_N(\theta_j))\) \((j = 0, \ldots, N - 1)\) in \(f'/f(z)\) (cf., equations (8.7)–(8.9)) has positive real part for \(z\) on \(C_u\).

Now note that

\[
-g(z(\phi)) = \frac{1}{2s_N(\theta_0) \sin \phi}, \quad \delta \leq \phi \leq \pi - \delta, \quad \text{(8.12)}
\]

is real and positive; we anticipate it to be the dominant term of the logarithmic derivative \(-f'/f(z)\). We will show that the real parts of each of the terms \(\frac{1}{z(\phi) - e_N(\theta_j)}\) (for \(\delta \leq \phi \leq \pi - \delta\) and \(j = 1, \ldots, N - 2\)) are bounded above by a suitably small multiple of \(-g(z(\phi))\).

**Lemma 8.2.** For all \(z\) with \(|z| < 1\) we have

\[
\max_{|\zeta| = 1} \text{Re} \frac{1}{z - \zeta} = \frac{1 - \text{Re}(z)}{1 - |z|^2}. \quad \text{(8.13)}
\]

This lemma can be proved by rewriting the equation \(|\zeta| = 1\) in the form

\[
\left| w + \frac{\bar{z}}{1 - |z|^2} \right| = \frac{1}{1 - |z|^2},
\]

in terms of the variable

\[
w = \frac{1}{z - \zeta},
\]

whence the result follows trivially.

**Lemma 8.3.** For \(N \geq 3, 0 \leq \theta_0 \leq 1, 0 < \phi < \pi\) and all \(\psi\) we have

\[
\text{Re} \frac{1}{z(\phi) - e_N(\psi)} \leq -\left( \frac{\pi \theta_0}{N} + 2\pi^2(7 + 4\sqrt{3})\frac{\theta_0^2}{N^2} \right) g(z(\phi)), \quad \text{(8.14)}
\]

First, since \(-g(z(\phi)) > 0\) for \(0 < \phi < \pi\) (cf., equation (8.12)), it suffices to prove the upper bound

\[
\left( \frac{\pi \theta_0}{N} + 2\pi^2(7 + 4\sqrt{3})\frac{\theta_0^2}{N^2} \right)
\]

for the quantity

\[
h(\theta_0; \phi, \psi) := \frac{-1}{g(z(\phi))} \text{Re} \frac{1}{z(\phi) - e_N(\psi)} = s_N(\theta_0) \sin \phi \text{Re} \frac{1}{z(\phi) - e_N(\psi)}. \quad \text{(8.15)}
\]

By Lemma 8.2,

\[
h(\theta_0; \phi, \psi) \leq \left( \frac{1 - \text{Re}(z(\phi))}{1 - |z(\phi)|^2} \right) s_N(\theta_0) \sin \phi.
\]
From Equation (8.10) it quickly follows that
\[ 1 - |z(\phi)|^2 = s_N(2\theta_0) \sin \phi, \quad \text{and} \quad \Re(z(\phi)) = c_N(\theta_0) - s_N(\theta_0) \sin \phi. \]
Thus,
\[ h(\theta_0; \phi, \psi) \leq \left( \frac{1 - c_N(\theta_0) + s_N(\theta_0) \sin \phi}{s_N(2\theta_0) \sin \phi} \right) \frac{s_N(\theta_0) \sin \phi}{1 - c_N(\theta_0) + s_N(\theta_0) \sin \phi} \]
\[ = \frac{1}{2} \tan \left( \frac{2\pi\theta_0}{N} \right) \sin \phi + \frac{1}{2} \left( \sec \left( \frac{2\pi\theta_0}{N} \right) - 1 \right) \]
\[ \leq \frac{1}{2} \tan \left( \frac{2\pi\theta_0}{N} \right) + \frac{1}{2} \left( \sec \left( \frac{2\pi\theta_0}{N} \right) - 1 \right). \] (8.17)

Let us now use the Taylor formulæ with remainder
\[ |\tan \theta - \theta| \leq \frac{\theta^2}{2} \sup_{|\vartheta| \leq \theta} |\tan'' \vartheta| \]
\[ |\sec \theta - 1| \leq \frac{\theta^2}{2} \sup_{|\vartheta| \leq \theta} |\sec'' \vartheta| \]
with \( \theta := 2\pi\theta_0/N \). Both \( |\tan'' \vartheta| \) and \( |\sec'' \vartheta| \) are even functions of \( \vartheta \), increasing with \( |\vartheta| \), so their respective suprema are bounded by \( |\tan''(2\pi/3)| = 14 \) and \( |\sec''(2\pi/3)| = 8\sqrt{3} \) (since \( \vartheta = \theta = 2\pi\theta_0/N \leq 2\pi/3 \), by the assumptions \( \theta_0 \leq 1 \) and \( N \geq 3 \)). The inequalities
\[ \tan \theta \leq \theta + 7\theta^2, \quad \text{and} \quad \sec \theta - 1 \leq 4\sqrt{3} \theta^2, \]
follow immediately. These inequalities together with (8.17) complete the proof of Lemma 8.3 upon setting \( \theta = 2\pi\theta_0/N \).

To complete the proof that \( D_\ell \) is contained in \( \Re w < 0 \), it remains to show that
\[ -\Re \left( \frac{f'}{f}(z(\phi)) \right) > 0 \quad \text{for} \quad \delta \leq \phi \leq \pi - \delta. \] (8.18)
But
\[ -\Re \left( \frac{f'}{f}(z(\phi)) \right) = -g(z(\phi)) - \sum_{j=1}^{N-2} \Re \frac{1}{z(\phi) - e_N(\theta_j)} \quad \text{since} \quad -g(z(\phi)) > 0 \]
\[ \geq -g(z(\phi)) \left[ 1 - (N - 2) \left( \frac{\pi\theta_0}{N} + 2\pi^2(7 + 4\sqrt{3}) \frac{\theta_0^2}{N^2} \right) \right] \quad \text{by Lemma 8.3} \]
\[ = -g(z(\phi)) \left[ 1 - \pi\theta_0 + U \theta_0 \frac{\theta_0}{N} + V \frac{\theta_0^2}{N} + W \frac{\theta_0^2}{N^2} \right], \]
say, for suitable absolute constants \( U, V, W \). As \( N \to \infty \), the last bracketed quantity above has the limit \( 1 - \pi\theta_0 \). As long as \( |\theta_0| \leq T < \frac{1}{2} \), there will exist \( N(T) \) such that said quantity is positive for \( N \geq N(T) \). This completes the proof of (8.18) and of the uniqueness of the
desired root \( z' = \zeta(\Theta_0) \) of \( f' \). The analyticity of \( \zeta \) as a function of \( \Theta_0 \) follows from the joint analyticity of the function \( f'(\Theta_0; z) \) in the variables \( \Theta_0 \) and \( z \) together with the formula

\[
\zeta(\Theta_0) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{zf''(\Theta_0; z)}{f'(\Theta_0; z)} \, dz. \tag{8.19}
\]

While \( \mathcal{C} \) depends on the choice of a fixed \( \epsilon \), it is clear that \( \epsilon \) can be chosen so that \( z' = \zeta(\Theta_0) \) is uniquely defined by the integral (8.19) for all \( \Theta_0 \) in any desired compact subset of the interior of \( \mathcal{P}_N^{(T)} \). This is enough to ensure that \( \zeta \) is analytic in the whole interior of \( \mathcal{P}_N^{(T)} \) and concludes the proof of Proposition 8.1. \( \square \)

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<table>
<thead>
<tr>
<th>No.</th>
<th>Title</th>
<th>Authors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>'CLOSED ORBITS AND UNIFORM S-INSTABILITY IN GEOMETRIC INVARIANT THEORY'</td>
<td>Michael Bate, Benjamin Martin, Gerhard Röhrle &amp; Rudolf Tange</td>
</tr>
<tr>
<td>2</td>
<td>'BUNDLES OF COLOURED POSETS AND A LERAY-SERRE SPECTRAL SEQUENCE'</td>
<td>Brent Everitt &amp; Paul Turner</td>
</tr>
<tr>
<td>3</td>
<td>'STATE-DEPENDENT FOSTER-LYAPUNOV CRITERIA FOR SUBGEOMETRIC CONVERGENCE OF MARKOV CHAINS'</td>
<td>Stephen Connor &amp; Gersende Fort</td>
</tr>
<tr>
<td>4</td>
<td>'INDEPENDENCE ALGEBRAS, BASIS ALGEBRAS AND SEMIGROUPS OF QUOTIENTS'</td>
<td>Victoria Gould</td>
</tr>
<tr>
<td>5</td>
<td>'RESTRICTION SEMIGROUPS AND INDUCTIVE CONSTELLATIONS'</td>
<td>Victoria Gould &amp; Christopher Hollings</td>
</tr>
<tr>
<td>6</td>
<td>'PERFECTION FOR POMONOIDS'</td>
<td>Victoria Gould &amp; Lubna Shaheen</td>
</tr>
<tr>
<td>7</td>
<td>'OPTIMAL CO-ADAPTED COUPLING FOR A RANDOM WALK ON THE HYPER-COMPLETE-GRAPH'</td>
<td>Stephen Connor</td>
</tr>
<tr>
<td>8</td>
<td>'BACKWARD UNIQUENESS AND THE EXISTENCE OF THE SPECTRAL LIMIT FOR SOME PARABOLIC SPDES'</td>
<td>Zdzislaw Brzezniak &amp; Misha Neklyudov</td>
</tr>
<tr>
<td>9</td>
<td>'THE GEOMETRY OF GENERALISED CHEEGER-GROMOLL METRICS'</td>
<td>Chris Wood, Michele Benyounes &amp; Eric Loubeau</td>
</tr>
<tr>
<td>10</td>
<td>'Multipartite Entanglement and Hypermatrices'</td>
<td>Anthony Sudbery &amp; Joseph Hilling</td>
</tr>
</tbody>
</table>
No. 11 ‘COMPLETE REDUCIBILITY AND SEPARABLE FIELD EXTENSIONS’

Michael Bate, Benjamin Martin & Gerhard Röhrle

No. 12 ‘A SCALE-ININVARIANT MODEL OF MARINE POPULATION DYNAMICS’

Gustav Delius & José Capitán

No. 13 ‘THE CLASSIFICATION OF HAMILTONIAN STATIONARY LAGRANGIAN TORI IN CP^2 BY THEIR SPECTRAL DATA’

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Ian McIntosh & Pascal Romon

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Chris Hughes, Eduardo Dueñez, David W. Farmer, Sara Froehlich, Francesco Mezzadri & Toan Phan