‘BISIMPLE INVERSE \(\omega\)-SEMIGROUPS OF LEFT I-QUOTIENTS’

Nassraddin Ghroda
BISIMPLE INVERSE $\omega$-SEMIGROUPS OF LEFT I-QUOTIENTS

N. GHRODA

Abstract. A subsemigroup $S$ of an inverse semigroup $Q$ is a left I-order in $Q$ if every element in $Q$ can be written as $a^{-1}b$ where $a, b \in S$ and $a^{-1}$ is the inverse of $a$ in the sense of inverse semigroup theory. If we insist on $a$ and $b$ being $R$-related in $Q$, then we say that $S$ is a straight left I-order in $Q$. We give necessary and sufficient conditions for a semigroup to be a left I-order in a bisimple inverse $\omega$-semigroup.

1. Introduction

Many definitions of semigroups of quotients have been proposed and studied. The first, that was specifically tailored to the structure of semigroups was introduced by Fountain and Petrich in [2], but was restricted to completely 0-simple semigroups of left quotients. This definition has been extended to the class of all semigroups [6]. The idea is that a subsemigroup $S$ of a semigroup $Q$ is a left order in $Q$ or $Q$ is a semigroup of left quotients of $S$ if every element of $Q$ can be written as $a^2b$ where $a, b \in S$ and $a^2$ is the inverse of $a$ in a subgroup of $Q$ and if, in addition, every square-cancellable element (an element $a$ of a semigroup $S$ is square-cancellable if $aH\ast a^2$) lies in a subgroup of $Q$. Semigroups of right quotients and right orders are defined dually. If $S$ is both a left order and a right order in a semigroup $Q$, then $S$ is an order in $Q$ and $Q$ is a semigroup of quotients of $S$. This definition and its dual were used in [6] to characterize semigroups which have bisimple inverse $\omega$-semigroups of left quotients.

On the other hand, Clifford [1] showed that from any right cancellative monoid $S$ with (LC) there is a bisimple inverse monoid $Q$ such that $Q = S^{-1}S$; that is, every element $q$ in $Q$ can be written as $a^{-1}b$ where $a, b \in S$ and $a^{-1}$ is the inverse of $a$ in $Q$ in the sense of inverse semigroup theory. By saying that a semigroup $S$ has the (LC) condition we mean that for any $a, b \in S$ there is an element $c \in S$ such that $Sa \cap Sb = Sc$. The author and Gould in [4] have extended Clifford’s work to a left ample semigroup with (LC) where they introduced the following definition of left I-orders in inverse semigroups:

Date: August 20, 2010.

Key words and phrases. bisimple inverse $\omega$-semigroup , I-quotients, I-order.
Let $Q$ be an inverse semigroup. A subsemigroup $S$ of $Q$ is a \textit{left I-order} in $Q$ or $Q$ is a semigroup of \textit{left I-quotients} of $S$, if every element in $Q$ can be written as $a^{-1}b$ where $a, b \in S$. The notions of \textit{right I-order} and \textit{semigroup of right I-quotients} are defined dually. If $S$ is both a left I-order and a right I-order in $Q$, we say that $S$ is an \textit{I-order} in $Q$ and $Q$ is a semigroup of \textit{I-quotients} of $S$. It is clear that, if $S$ a left order in an inverse semigroup $Q$, then it is certainly a left I-order in $Q$; however, the converse is not true (see for example [4] Example 2.2).

A left I-order $S$ in an inverse semigroup $Q$ is \textit{straight left I-order} if every element in $Q$ can be written as $a^{-1}b$ where $a, b \in S$ and $a R b$ in $Q$; we also say that $Q$ is a \textit{semigroup of straight left I-quotients} of $S$. If $S$ is straight in $Q$, we have the advantage of controlling products in $Q$. Many left I-orders are straight, such as left I-orders in primitive inverse semigroups. In the case where $S$ is a straight left I-order in $Q$, the uniqueness problem has been solved [4], that is, the author and Gould have given necessary and sufficient conditions for a left I-order to have a unique semigroup of a left I-quotients.

In [5] it was shown that if $H$ is a congruence on a regular semigroup $Q$, then every left order $S$ in $Q$ is straight. To prove this, Gould uses the fact that $S$ intersects every $H$-class of $Q$. Since $H$ is congruence on any bisimple inverse $\omega$-semigroup, any left order $S$ in such a semigroup must be straight. In the case of left I-orders we show that if $S$ is a left I-order in a bisimple inverse $\omega$-semigroup $Q$, then $S$ intersects every $L$-class of $Q$ and we use this to show that $S$ is straight in $Q$.

The main aim of this article is to study semigroups which have bisimple inverse $\omega$-semigroups of left I-quotients. After giving some preliminaries in Section [2] in Section [3] we extend the result in [6], by introducing the main theorem in this article, which gives necessary and sufficient conditions for a semigroup to be a left I-order in a bisimple inverse $\omega$-semigroup.
2. Preliminaries

Throughout this article we shall follow the terminology and notation of [1]. The set of all non-negative integers will be denoted by \( \mathbb{N}^0 \). Let \( \mathcal{R}, \mathcal{L}, \mathcal{H} \) and \( \mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} \) be the usual Green’s relations. A semigroup \( S \) is bisimple if it consists of a single \( \mathcal{D} \)-class.

For any semigroup \( Q \) with the set of idempotents \( E \) we define a partial ordering \( \leq \) on \( E \) by the rule that \( e \leq f \) if and only if \( ef = fe = e \). A bisimple inverse \( \omega \)-semigroup is a bisimple inverse semigroup whose idempotents form an \( \omega \)-chain; that is, \( E(S) = \{ e_m : m \in \mathbb{N}^0 \} \) where \( e_0 \geq e_1 \geq e_2 \geq \ldots \). Thus if \( S \) is a bisimple inverse \( \omega \)-semigroup, on \( E \) we have \( e_m \leq e_n \) if and only if \( m \geq n \).

Reilly [8] determined the structure of all bisimple inverse \( \omega \)-semigroups as follows:

Let \( G \) be a group and let \( \theta \) be an endomorphism of \( G \). Let \( S(G, \theta) \) be the semigroup on \( \mathbb{N}^0 \times G \times \mathbb{N}^0 \) with multiplication

\[
(m, g, n)(p, h, q) = (m - n + t, (g\theta^{t-n})(h\theta^{t-p}), q - p + t)
\]

where \( t = \max\{n, p\} \) and \( \theta^0 \) is interpreted as the identity map of \( G \). As was shown in [8] (Cf [7]), \( S(G, \theta) \) is a bisimple inverse \( \omega \)-semigroup and every bisimple inverse \( \omega \)-semigroup is isomorphic to \( S(G, \theta) \). In the case where \( G \) is trivial, then \( S(G, \theta) = B \) where \( B \) the bicyclic semigroup, we identify \( B \) with \( \mathbb{N}^0 \times \mathbb{N}^0 \). The set of idempotents of \( S(G, \theta) \) is \( \{ (m, 1, m) : m \in \mathbb{N}^0 \} \) and for any \( (m, g, n) \) in \( S(G, \theta) \),

\[
(m, g, n)^{-1} = (n, g^{-1}, m).
\]

For any \( (m, a, n), (p, b, q) \in S(G, \theta) \),

\[
(m, a, n) \mathcal{R} (p, b, q) \text{ if and only if } m = p,
\]

\[
(m, a, n) \mathcal{L} (p, b, q) \text{ if and only if } n = q,
\]

and, consequently,

\[
(m, a, n) \mathcal{H} (p, b, q) \text{ if and only if } m = p \text{ and } n = q.
\]

If \( Q \) is a bisimple inverse \( \omega \)-semigroup, let \( R_n \) (\( L_n \)) denote the \( \mathcal{R} \)-class (\( \mathcal{L} \)-class) of \( Q \) containing the idempotent \( e_n = (n, 1, n) \). From the above,

\[
R_m = \{(m, a, n) : a \in G, n \in \mathbb{N}^0 \},
\]

\[
L_n = \{(m, a, n) : a \in G, m \in \mathbb{N}^0 \}.
\]

Clearly,

\[
H_{m,n} = R_m \cap L_n = \{(m, a, n) : a \in G \}
\]

\[
= \{ q \in Q : qq^{-1} = e_m, q^{-1}q = e_n \}.
\]
and from the multiplication in $S(G, \theta)$,
\[ H_{m,n}H_{p,q} \subseteq H_{m-n+t,q-p+t}, \]
where $t = \max\{n, p\}$.

Let $S$ be any semigroup such that there is a homomorphism $\varphi : S \rightarrow B$. We define functions $l, r : S \rightarrow \mathbb{N}^0$ by $a\varphi = (r(a), l(a))$. We also put $H_{i,j} = (i, j)\varphi^{-1}$, so that $S$ is a disjoint union of subsets of the $H_{i,j}$ and
\[ H_{i,j} = \{a \in S : r(a) = i, l(a) = j\}. \]

From the above, $\mathcal{H}$ is a congruence on any bisimple inverse $\omega$-semigroup $Q$. Moreover there is a homomorphism $\varphi : Q \rightarrow B$ given by $(m, p, n)\varphi = (m, n)$ which is surjective with $\text{Ker}\varphi = \mathcal{H}$ so $Q/\mathcal{H} \cong B$ where $B$ is the bicyclic semigroup. As above we will index $\mathcal{H}$ in $Q$ by putting $H_{i,j} = (i, j)\varphi^{-1}$.

Let $S$ be a left $I$-order in $Q$. Let $\varphi = \varphi|_S$, then $\varphi$ is a homomorphism from $S$ to $B$. Unfortunately, this homomorphism is not surjective in general, since $S$ need not intersect every $\mathcal{H}$-class of $Q$. But we can as above index the elements of $S$.

In [3] it was shown that, if a semigroup $S$ is a left $I$-order in a bicyclic semigroup $B$, then $S$ intersects every $L$-class of $B$. Moreover, it is straight. In fact, this is true for any left $I$-order in a bisimple inverse $\omega$-semigroup, as we will see in the next lemmas.

**Lemma 2.1.** If a semigroup $S$ is a left $I$-order in a bisimple inverse $\omega$-semigroup $Q$, then $S \cap L_n \neq \emptyset$ for all $n \in \mathbb{N}^0$.

**Proof.** Let $p \in H_{n,n}$, then $p = a^{-1}b$ for some $a, b \in S$ with $a \in H_{i,j}$ and $b \in H_{k,l}$. Hence
\[ p = a^{-1}b \in H_{i,j}H_{k,l} \subseteq H_{j-i+\max(i,k),l-k+\max(i,k)}, \]
and so $n = j - i + \max(i, k) = l - k + \max(i, k)$. So, as $\max(i, k) = i$ or $k$, either $n = j$ or $n = l$. Hence $S \cap L_n \neq \emptyset$. \qed

In [6] it was shown that if $S$ a left order in a bisimple inverse $\omega$-semigroup $Q$, then it is straight. The following lemma extends this to the left $I$-order in a bisimple inverse $\omega$-semigroup.

**Lemma 2.2.** If a semigroup $S$ is a left $I$-order in a bisimple inverse $\omega$-semigroup $Q$, then $S$ is straight.

**Proof.** Let $(h, q, k) \in Q$, then $(h, q, k) = (i, a, j)^{-1}(t, b, s) = (j, a^{-1}, i)(t, b, s)$ for some $(i, a, j), (t, b, s) \in S$. Let $n = \max\{i, t\}$; since $S \cap L_n \neq \emptyset$, by Lemma 2.1.
there exist \((u,c,n) \in S \cap L_n\) and hence \((u,c,n)^{-1}(u,c,n) = (n,1,n)\), so that \((n,1,n)R(t,b,s)\) or \((n,1,n)R(i,a,j)\). In both cases, we have
\[
(h,q,k) = (i,a,j)^{-1}(n,1,n)(t,b,s) = (i,a,j)^{-1}(u,c,n)^{-1}(u,c,n)(t,b,s) = ((u,c,n)(i,a,j))^{-1}((u,c,n)(t,b,s)).
\]
It is clear that \((u,c,n)(i,a,j)R(u,c,n)(t,b,s)\). Hence \(S\) is straight. \(\square\)

**Proposition 2.3.** Let \(Q\) be an inverse semigroup and \(q = a^{-1}b\) with \(aRb\), then \(a^{-1}RqLb\).

The following corollaries are clear.

**Corollary 2.4.** Let \(Q\) be an inverse semigroup. If \(a^{-1}b,c^{-1}d \in Q\) where \(aRb\) and \(cRd\), then
(i) \(a^{-1}bRc^{-1}d \iff a^{-1}a = c^{-1}c\);
(ii) \(a^{-1}bLc^{-1}d \iff b^{-1}b = d^{-1}d\).

**Corollary 2.5.** Let \(Q\) be a bisimple inverse \(\omega\)-semigroup, then
(i) \((m,a,n)^{-1}(m,b,t)R(i,c,j)^{-1}(i,d,k)\) if and only if \(n = j\);
(ii) \((m,a,n)^{-1}(m,b,t)L(i,c,j)^{-1}(i,d,k)\) if and only if \(t = k\).

### 3. The main theorem

This section is entirely devoted to proving Theorem 3.1 which gives a characterisation of semigroups which have a bisimple inverse \(\omega\)-semigroup of left I-quotients.

**Theorem 3.1.** A semigroup \(S\) is a left I-order in a bisimple inverse \(\omega\)-semigroup \(Q\) if and only if \(S\) satisfies the following conditions:
(A) There is a homomorphism \(\varphi : S \to B\) such that \(S\varphi\) is a left I-order in \(B\);
(B) For \(x,y,a \in S\),
\[
(i) \ l(x),l(y) \geq r(a) \text{ and } xa = ya \text{ implies } x = y,
(ii) \ r(x),r(y) \geq l(a) \text{ and } ax = ay \text{ implies } x = y.
\]
(C) For any \(b,c \in S\), there exist \(x,y \in S\) such that \(xb = yc\) where
\[
x \in H_{r(x),r(b)-l(b)+\max\{l(b),l(c)\}}, y \in H_{r(y),r(c)-l(c)+\max\{l(b),l(c)\}}.
\]

**Proof.** Let \(S\) be a left I-order in a bisimple inverse \(\omega\)-semigroup \(Q\). For condition (A), since \(S\) is a left I-order in \(Q\) and there is a homomorphism \(\overline{\varphi} : Q \to B\) given by
\[
(m,p,n)\overline{\varphi} = (m,n),
\]
we can restrict \( \varphi \) on \( S \) to get a homomorphism \( \varphi \) from \( S \) to \( B \). Let \((i, j) \in B\), then there is an element \( q \) in \( Q \) such that \( q \in H_{i,j} \) for some \( i, j \in \mathbb{N}^0 \). Put \( q = a^{-1}b \) for some \( a, b \in S \) with \( a \# b \) in \( Q \), so that \( r(a) = r(b) \). Hence

\[
q \in H_{l(a),r(a)} H_{r(a),l(b)} \subseteq H_{l(a),l(b)},
\]
then

\[
(i, j) = (l(a), l(b)) = (r(a), l(a))^{-1}(r(b), l(b)) = (a \varphi)^{-1}(b \varphi).
\]

To see that \((B)(i)\) holds, suppose that \( x, y, a \in S \) where \( l(x), l(y) \geq r(a) \) and \( xa = ya \). Since \( a^{-1} \in H_{l(a),r(a)} \) and \( xaa^{-1} = yaa^{-1} \), that is, \( xe_{r(a)} = ye_{r(a)} \), and \( r(a) \leq l(x), l(y) \), then we have \( e_{l(x)} e_{l(y)} \leq e_{r(a)} \). Hence \( x e_{l(x)} e_{r(a)} = y e_{l(y)} e_{r(a)} \) and so \( x = x e_{l(x)} = y e_{l(y)} = y \).

\((B)(ii)\) Similar to \((B)(i)\).

Finally, we consider \((C)\). Let \( b, c \in S \), then \( bc^{-1} \in Q \) and

\[
bc^{-1} \in H_{r(b),l(b)} H_{l(c),r(c)} \subseteq H_{r(b) - l(b) + \max(l(b), l(c)), r(c) - l(c) + \max(l(b), l(c))}.
\]

Since \( S \) is a straight left I-order in \( Q \), then \( bc^{-1} = x^{-1}y \) where \( x \# y \) for some \( x, y \in S \), and by Lemma 2.6 in [4], \( xb = yc \). From \( bc^{-1} = x^{-1}y \) we have

\[
H_{r(b) - l(b) + \max(l(b), l(c)), r(c) - l(c) + \max(l(b), l(c))} = H_{l(x), l(y)},
\]
so that \( l(x) = r(b) - l(b) + \max(l(b), l(c)) \) and \( l(y) = r(c) - l(c) + \max(l(b), l(c)) \).

Conversely, we suppose that \( S \) satisfies conditions \((A)\), \((B)\) and \((C)\). Now, our aim is to construct via equivalence classes of order pairs of elements of \( S \) an inverse semigroup \( Q \), which is a semigroup of straight left I-quotients of \( S \). First, we let

\[
\Sigma = \{(a, b) \in S \times S : r(a) = r(b)\}
\]
and on \( \Sigma \) we define the relation \( \sim \) as follows:

\( (a, b) \sim (c, d) \iff \) there are elements \( x, y \) in \( S \) such that \( xa = yc \) and \( xb = yd \)

where \( l(x) = r(a), l(y) = r(c) \) and \( r(x) = r(y) \). Notice that if \( (a, b) \sim (c, d) \), then \( l(a) = l(c) \) and \( l(b) = l(d) \).

**Lemma 3.2.** *The relation \( \sim \) is an equivalence.*

**Proof.** It is clear that \( \sim \) is symmetric. Let \((a, b) \in \Sigma\), by \((C)\) for any \( a \in S \) there exist \( x \in S \) with \( l(x) = r(a) \), so that \( \sim \) is reflexive.
Suppose that \((a, b) \sim (c, d) \sim (p, q)\). Then there are elements \(x, y, \bar{x}, \bar{y}\) in \(S\) with
\[
xa = yc \text{ and } xb = yd,
\]
\[
\bar{xc} = \bar{yp} \text{ and } \bar{xd} = \bar{yq},
\]
where
\[
r(x) = r(y), l(x) = r(a), l(y) = r(c),
\]
and
\[
r(\bar{x}) = r(\bar{y}), l(\bar{x}) = r(c), l(\bar{y}) = r(p).
\]
By Condition \((C)\), for \(y, \bar{x}\) there exist \(s, t \in S\) such that \(s \bar{x} = ty\) where
\[
s \in H_{r(s), r(\bar{x}) - l(\bar{x})} \max \{l(\bar{x}), l(y)\}, t \in H_{r(s), r(y) - l(y)} \max \{l(\bar{x}), l(y)\}.
\]
Since \(l(\bar{x}) = r(c) = l(y)\), then \(l(s) = r(\bar{x})\) and \(l(t) = r(y) = r(x)\). Now,
\[
txa = tyc = s\bar{xc} = s\bar{yp},
\]
and
\[
txb = tyd = s\bar{xd} = s\bar{yq}.
\]
Hence \(txa = s\bar{yp}\) and \(txb = s\bar{yq}\) where \(tx \in H_{r(s), r(a)}\), \(s\bar{y} \in H_{r(s), r(p)}\). We have
\[
l(tx) = r(a), l(s\bar{y}) = r(p) \text{ and } r(tx) = r(s\bar{y}),
\]
that is, \((a, b) \sim (p, q)\). Thus \(\sim\) is transitive. \(\square\)

We write the \(\sim\)-equivalence class of \((a, b)\) as \([a, b]\) and denote by \(Q\) the set of all \(\sim\)-equivalence classes of \(\Sigma\). If \([a, b], [c, d] \in Q\), then by \((C)\) for \(b\) and \(c\) there exist \(x, y\) such that \(xb = yc\) where
\[
x \in H_{r(x), r(a) - l(b)} \max \{l(b), l(c)\}, y \in H_{r(x), r(c) - l(c)} \max \{l(b), l(c)\}
\]
and it is easy to see that
\[
r(xa) = r(xb) = r(yc) = r(yd) = r(x) = r(y)
\]
and we deduce that \([xa, yd] \in Q\). Define a multiplication on \(Q\) by
\[
[a, b][c, d] = [xa, yd] \text{ where } xb = yc
\]
and \(x \in H_{r(x), r(b) - l(b)} \max \{l(b), l(c)\}, y \in H_{r(x), r(c) - l(c)} \max \{l(b), l(c)\} .
\]

**Lemma 3.3.** The given multiplication is well defined.

**Proof.** Suppose that \([a_1, b_1] = [a_2, b_2]\) and \([c_1, d_1] = [c_2, d_2]\). Then there are elements \(x_1, x_2, y_1, y_2\) in \(S\) such that
\[
x_1a_1 = x_2a_2, \\
x_1b_1 = x_2b_2, \\
y_1c_1 = y_2c_2, \\
y_1d_1 = y_2d_2,
\]
where
\[ l(x_1) = r(a_1), \quad l(x_2) = r(a_2), \quad r(x_1) = r(x_2) \]
and
\[ l(y_1) = r(c_1), \quad l(y_2) = r(c_2), \quad r(y_1) = r(y_2). \]
Note that, consequently,
\[ l(a_1) = l(a_2), l(b_1) = l(b_2), l(c_1) = l(c_2) \text{ and } l(d_1) = l(d_2). \]
Then
\[ [a_1, b_1][c_1, d_1] = [xa_1, yd_1] \text{ where } xb_1 = yc_1 \]
and \[ x \in H_{r(x), r(b_1) - l(b_1) + \max(l(b_1), l(c_1))} \times \in H_{r(x), r(c_1) - l(c_1) + \max(l(b_1), l(c_1))} \]
Also,
\[ [a_2, b_2][c_2, d_2] = [\bar{x}a_2, \bar{y}d_2] \text{ where } \bar{x}b_2 = \bar{y}c_2 \]
and \[ \bar{x} \in H_{r(\bar{x}), r(b_2) - l(b_2) + \max(l(b_2), l(c_2))} \times \in H_{r(\bar{x}), r(c_2) - l(c_2) + \max(l(b_2), l(c_2))} \].
We have to show that \([xa_1, yd_1] = [\bar{x}a_2, \bar{y}d_2].\]

Before completing the proof of Lemma 3.3 we present the following lemma.

**Lemma 3.4.** Let \(a_1, a_2, b_1, b_2 \in S\) be such that
\[ r(a_1) = r(b_1), r(a_2) = r(b_2) \]
and suppose that \(x_1, x_2, w_1, w_2 \in S\) are such that
\[ x_1a_1 = x_2a_2, \quad x_1b_1 = x_2b_2, \quad w_1a_1 = w_2a_2 \]
where \(r(x_1) = r(x_2), l(x_1) = r(a_1), l(x_2) = r(a_2)\) and \(r(w_1) = r(w_2).\) Then \(w_1b_1 = w_2b_2.\)

**Proof.** Let \(a_1, a_2, b_1, b_2, x_1, x_2, w_1, w_2\) exist as given. Note that consequently \(l(a_1) = l(a_2)\) and \(l(b_1) = l(b_2).\) By (C) for \(w_1, x_1\) there exist \(x, y \in S\) such that \(xw_1 = yx_1\) where
\[ x \in H_{r(x), r(w_1) - l(w_1) + \max(l(w_1), l(x_1))}, \quad y \in H_{r(x), r(x_1) - l(x_1) + \max(l(w_1), l(x_1))}. \]
Then \(xw_1a_1 = yx_1a_1,\) and
\[ xw_2a_2 = xw_2a_1 = yx_1a_1 = yx_2a_2. \]
Now,
\[ xw_2 \in H_{r(x), l(w_2) - l(w_1) + \max(l(w_1), l(x_1))}, \quad yx_2 \in H_{r(x), l(x_2) - l(x_1) + \max(l(w_1), l(x_1))} \]
and as \(l(x_1) = r(a_1)\) and \(l(x_2) = r(a_2),\) we have
\[ l(yx_2) = r(a_2) - r(a_1) + \max(l(w_1), r(a_1)) \geq r(a_2) \]
and
\[ l(xw_2) = l(w_2) - l(w_1) + \max(l(w_1), r(a_1)). \]
As \( wx_2a_2 = yx_2a_2 \), then in order to use Condition \((B)(i)\), we have to show that \( l(xw_2) \geq r(a_2) \). Since \( w_1a_1 = w_2a_2 \),
\[
 r(w_1) - l(w_1) + \max(l(w_1), r(a_1)) = r(w_1) - l(w_2) + \max(l(w_2), r(a_2)) \tag{3.1}
\]
so that
\[
l(w_2) - l(w_1) + \max(l(w_1), r(a_1)) = \max(l(w_2), r(a_2)) \geq r(a_2)
\]
as desired. Therefore by condition \((B)(i)\), \( xw_2 = yx_2 \). Since \( xw_1 = yx_1 \) and \( x_1b_1 = x_2b_2 \) we have
\[
xw_1b_1 = yx_1b_1 = yx_2b_2 = xw_2b_2.
\]
Once we show that \( r(w_1b_1), r(w_2b_2) \geq l(x) \), by \((B)(ii)\) we have \( w_1b_1 = w_2b_2 \). Now,
\[
w_1b_1 \in H_{r(w_1) - l(w_1) + \max(l(w_1), r(b_1))} \cdot l(b_1) - r(b_1) + \max(l(w_1), r(b_1))
\]
and
\[
w_2b_2 \in H_{r(w_1) - l(w_2) + \max(l(w_2), r(b_2))} \cdot l(b_2) - r(b_2) + \max(l(w_2), r(b_2))
\]
so that
\[
r(w_1b_1) = r(w_1) - l(w_1) + \max(l(w_1), r(a_1)) \quad \text{as} \quad l(x_1) = r(a_1) = r(b_1)
\]
\[
= r(w_1) - l(w_1) + \max(l(w_1), l(x_1)) = l(x)
\]
and
\[
r(w_2b_2) = r(w_1) - l(w_2) + \max(l(w_2), r(a_2)) \quad \text{as} \quad r(b_2) = r(a_2)
\]
\[
= r(w_1) - l(w_1) + \max(l(w_1), r(a_1)) \quad \text{by (3.1)}
\]
\[
= r(w_1) - l(w_1) + \max(l(w_1), l(x_1)) \quad l(x_1) = r(a_1)
\]
\[
= l(x).
\]
The proof of the Lemma is complete. \( \square \)

Returning to the proof of Lemma \( \textbf{3.3} \) by \( (C) \) for \( xa_1 \) and \( \bar{x}a_2 \) there exist \( w, \bar{w} \) such that \( wxa_1 = \bar{w}\bar{x}a_2 \) where
\[
w \in H_{r(w), r(xa_1) - l(xa_1) + \max(l(xa_1), l(\bar{x}a_2))} \quad \text{and} \quad \bar{w} \in H_{r(w), r(\bar{x}a_2) - l(\bar{x}a_2) + \max(l(\bar{x}a_1), l(\bar{x}a_2))}.
\]
Using the fact that \( l(b_1) = l(b_2), l(c_1) = l(c_2) \) and \( l(a_1) = l(a_2) \), it is easy to see that \( l(xa_1) = l(\bar{x}a_2) \). Therefore
\[
l(w) = r(xa_1) = r(x) \quad \text{and} \quad l(\bar{w}) = r(\bar{x}a_2) = r(\bar{x}).
\]
Hence \( r(wx) = r(w) = r(\bar{w}) = r(\bar{w}\bar{x}). \)

Now, \( x_1a_1 = x_2a_2, x_1b_1 = x_2b_2 \) and \( wxa_1 = \bar{w}\bar{x}a_2 \), so that by Lemma \( \textbf{3.4} \) we have \( wxb_1 = \bar{w}\bar{xb}_2 \).
We also have $xb_1 = yc_1$ and $xb_2 = yc_2$, and so $wyc_1 = wyc_2$. Thus
\[ y_1c_1 = y_2c_2, y_1d_1 = y_2d_2 \text{ and } wyc_1 = wyc_2. \]
Since $r(wy) = r(\bar{w}y)$, by using the above lemma again we have $wyd_1 = \bar{w}yd_2$.
Hence $[xa_1, yd_1] = [\bar{x}a_2, \bar{y}d_2]$. This completes the proof of Lemma 3.3.

The next lemma is useful in verifying that the given multiplication is associative. The proof follows immediately from the fact that $l(ab) \geq l(b)$, $l(de) \geq l(e)$, and (B)(i).

**Lemma 3.5.** Let $a, b, c, d, e \in S$. If $abc = dec$ and $l(b) \geq r(c), l(e) \geq r(c)$, then $ab = de$.

**Lemma 3.6.** The given multiplication is associative.

**Proof.** Let $[a, b], [c, d], [p, q] \in Q$. Then by using the definition of multiplication in $Q$ we have
\[ ([a, b][c, d])[p, q] = [xa, yd][p, q] \text{ where } xb = yc \]
and $x \in H_{r(x), r(b) - l(b) + \max(l(b), l(c))}, y \in H_{r(x), r(c) - l(c) + \max(l(b), l(c))}$ for some $x, y \in S$ and then
\[ ([a, b][c, d])[p, q] = [wxa, \bar{w}q] \text{ where } wyd = \bar{w}p \]
and $w \in H_{r(w), r(yd) - l(yd) + \max(l(yd), l(p))}, \bar{w} \in H_{r(w), r(p) - l(p) + \max(l(yd), l(p))}$ for some $w, \bar{w} \in S$. Similarly,
\[ [a, b][\{c, d\}[p, q]] = [a, b][\bar{x}c, \bar{y}q] \text{ where } \bar{x}d = \bar{y}p \]
and $\bar{x} \in H_{r(\bar{x}), r(d) - l(d) + \max(l(d), l(p))}, \bar{y} \in H_{r(\bar{x}), r(p) - l(p) + \max(l(d), l(p))}$, and then
\[ [a, b][\{c, d\}[p, q]] = [za, z\bar{y}q] \text{ where } zb = z\bar{x}c \]
and $z \in H_{r(z), r(b) - l(b) + \max(l(b), l(xc))}$ and $\bar{z} \in H_{r(z), r(\bar{x}c) - l(\bar{x}c) + \max(l(b), l(xc))}$.
To complete our proof we have to show that $[wxa, \bar{w}q] = [za, z\bar{y}q]$. That is, we need to show that there are $t, h \in S$ such that $twxa = hza$ and $twq = h\bar{y}q$ with
\[ r(t) = r(h), l(t) = r(wxa) \text{ and } l(h) = r(za). \]
By Condition (C) for $wx$, there exist $h, t \in S$ such that $twx = hz$ where
\[ t \in H_{r(t), r(wx) - l(wx) + \max(l(wx), l(z))}, h \in H_{r(t), r(z) - l(z) + \max(l(wx), l(z))}, \]
and so $twxa = hza$ and $twxb = hzb$. Since $xb = yc$ and $zb = \bar{z}xc$ we have $twyc = h\bar{z}xc$. But
\[ l(y) = r(c) - l(c) + \max(l(b), l(c)) \geq r(c) \]
and
\[l(x) = r(d) - l(d) + \max(l(d), l(p)) = r(c) - l(d) + \max(l(d), l(p)) \geq r(c).\]

By Lemma 3.5 we have \(twy = h\bar{z}\bar{x}\) and so \(twyd = h\bar{z}\bar{x}d\). Now, \(wyd = \bar{w}p\) and \(\bar{xd} = \bar{yp}\), so that \(t\bar{w}p = h\bar{z}\bar{y}p\). But
\[l(\bar{w}) = r(p) - l(p) + \max(l(yd), l(p)) \geq r(p)\]
and
\[l(\bar{y}) = r(p) - l(p) + \max(l(d), l(p)) \geq r(p),\]
so that by Lemma 3.5 \(t\bar{w} = h\bar{z}\bar{y}\). Hence \(t\bar{w}q = h\bar{z}\bar{y}q\). It remains to prove that \(l(t) = r(wx)a\) and \(l(h) = r(za)\).

Since
\[l(t) = r(wx) - l(wx) + \max(l(wx), l(z))\]
and
\[l(h) = r(z) - l(z) + \max(l(wx), l(z)).\]
Calculating, we have
\[r(wx) = r(w) \quad (3.2)\]
\[l(wx) = l(x) - l(yd) + \max(l(p), l(yd)) \quad (3.3)\]
\[l(z) = r(b) - l(b) + \max(l(b), l(\bar{xc})) \quad (3.4)\]
and
\[l(\bar{xc}) = l(c) - l(d) + \max(l(d), l(p)) \quad (3.5)\]
\[l(yd) = l(d) - l(c) + \max(l(b), l(c)) \quad (3.6)\]
Since \(r(wx) = r(wx)a\) and \(r(z) = r(za)\), once we show that \(l(z) = l(wx)\), we will have
\[l(t) = r(wx) = r(wx)a\) and \(l(h) = r(z) = r(za)\).
It is convenient to consider separately two cases.

Case(i): \(l(c) \geq l(b)\). We have \(l(yd) = l(d)\) and \(l(x) = r(b) - l(b) + l(c)\). If \(l(d) \geq l(p)\), then from (3.5) we have \(l(\bar{xc}) = l(c)\). From (3.3) and (3.4),
\[l(wx) = l(x) = r(b) - l(b) + l(c) = l(z).\]
If, on the other hand, \(l(d) \leq l(p)\), then \(l(\bar{xc}) = l(c) - l(d) + l(p)\). From (3.3) and (3.4),
\[l(wx) = l(x) - l(d) + l(p)\]
and
\[l(z) = r(b) - l(b) + \max(l(b), l(c) - l(d) + l(p))\]
Since \(l(c) \geq l(b)\) and \(l(d) \leq l(p)\), then \(l(c) - l(d) + l(p) \geq l(b)\). Thus
If, on the other hand, \( l(c) \leq l(b) \). We have \( l(yd) = l(d) - l(c) + l(b) \) and \( l(x) = r(b) \). If \( l(d) \geq l(p) \), then \( l(\bar{x}c) = l(c) \). From (3.3) and (3.4),
\[
    l(wx) = l(x) - l(d) + l(c) - l(b) + \max(l(p), l(d) - l(c) + l(b))
\]
and
\[
    l(z) = r(b) - l(b) + \max(l(b), l(c)) = r(b) = l(x).
\]
Since \( l(d) \geq l(p) \) and \( l(c) \leq l(b) \) we have \( l(d) - l(c) + l(b) \geq l(d) \geq l(p) \). Then \( l(wx) = l(x) \). Hence \( l(z) = l(x) = l(wx) \).

If, on the other hand, \( l(d) \leq l(p) \), then from (3.5) we have \( l(\bar{x}c) = l(c) - l(d) + l(p) \). From (3.3) and (3.4),
\[
    l(wx) = l(x) - l(d) + l(c) - l(b) + \max(l(p), l(d) - l(c) + l(b))
\]
and
\[
    l(z) = r(b) - l(b) + \max(l(b), l(c) - l(d) + l(p)).
\]
Once again, there are two cases. If \( l(c) - l(d) + l(p) \geq l(b) \), then
\[
    l(p) \geq l(d) - l(c) + l(b)
\]
and so
\[
    l(wx) = l(x) - l(d) + l(c) - l(b) + l(p)
    = r(b) - l(d) + l(c) - l(b) + l(p)
    = l(z).
\]
If, on the other hand, \( l(c) - l(d) + l(p) \leq l(b) \), then \( l(p) \leq l(d) - l(c) + l(b) \). Hence \( l(wx) = l(x) = r(b) = l(z) \).

This completes the proof of the lemma. \( \square \)

Now we aim to show that \( Q \), which we have constructed, is a semigroup of left I-quotients of \( S \). First we show that \( S \) is embedded in \( Q \).

Let \( a \in S \). Then \( a \in H_{r(a), l(a)} \) and as seen earlier, there exist \( x \in S \) with \( l(x) = r(a) \). Then \( xa \in H_{r(x), l(a)} \) and \([x, xa] \in Q\). If \( y \in S \) with \( l(y) = r(a) \), then \( ya \in H_{r(y), l(a)} \) and again \([y, ya] \in Q\). By (C) there exist \( s, t \in S \) with \( sx = ty \) (and so \( sxa = tya \)), where \( s \in H_{r(s), r(x)} \), \( t \in H_{r(s), r(y)} \). Hence \([x, xa] = [y, ya] \). There is therefore a well-defined mapping \( \theta : S \rightarrow Q \) defined by \( a \theta = [x, xa] \) where \( x \in H_{r(x), r(a)} \).

**Lemma 3.7.** The semigroup \( S \) is embedded in \( Q \).
**Proof.** Suppose that \( a\theta = b\theta \), that is, \( [x, xa] = [y, yb] \) where \( x \in H_{r(x), r(a)} \) and \( y \in H_{r(y), r(b)} \), then there exist \( s, t \in S \) such that \( sx = ty \) and \( sx = tyb \) where \( l(s) = r(x), l(t) = r(y) \) and \( r(s) = r(t) \). We claim that \( a = b \).

Since \( sx = tyb = sx \), once we show that \( r(a), r(b) \geq l(sx) \) we can use (B)(ii) to get \( a = b \). Now, it is easy to see that \( sx \in H_{r(s), r(a)} \) and \( ty \in H_{r(t), r(b)} \) and so \( l(sx) = r(a) \) and \( l(ty) = r(b) \). But \( sx = ty \), so that \( r(a) = r(b) = l(sx) \). Hence \( a = b \) and so \( \theta \) is 1-1, our claim is established.

To show that \( \theta \) is a homomorphism, let \( a\theta = [x, xa] \) and \( b\theta = [y, yb] \) where \( x \in H_{r(x), r(a)} \) and \( y \in H_{r(y), r(b)} \). Then
\[
(a\theta)(b\theta) = [x, xa][y, yb] = [wx, \bar{w}yb] \quad \text{where} \quad wxa = \bar{w}y
\]
and \( w \in H_{r(w), r(xa) - l(xa) + \max(l(xa), l(y))} \). Hence
\[
(a\theta)(b\theta) = [wx, wxab].
\]
Notice that
\[
(b\theta)(a\theta) = [y, yb][x, xa] = [l(a), l(xa)] = [l(a), r(xa) + \max(l(xa), l(y))]
\]
so that \( w \in H_{r(w), r(a) - l(a) + \max(l(a), r(b))} \). Then
\[
wxa = H_{r(w), r(xa) - l(xa) + \max(l(xa), l(y))} = H_{r(w), r(ab)}.
\]
It follows that \( (ab)\theta = [wx, wxab] = a\theta b\theta \). \( \square \)

The main purpose of the following is to show that \( Q \) is a bisimple inverse \( \omega \)-semigroup and \( S \) is a left I-order in \( Q \). First we need the following simple but useful lemma.

**Lemma 3.8.** Let \( [a, b] \in Q \). Then \( [a, b] = [xa, xb] \) for any \( x \in S \) with \( l(x) = r(a) \).

**Proof.** It is clear that \( r(xa) = r(x) = r(xb) \), so that \( [xa, xb] \in Q \). By (C) for \( a \) and \( xa \) there exist \( t, z \in S \) such that \( ta = zxa \) where
\[
t \in H_{r(t), r(a) - l(a) + \max(l(a), l(xa))}, \quad z \in H_{r(t), r(xa) - l(xa) + \max(l(a), l(xa))}.
\]
Since \( l(xa) = l(a) \) and \( r(xa) = r(x) \), we have \( l(t) = r(a) \) and \( l(z) = r(xa) = r(x) \). Also, \( l(zx) = r(a) \). Hence by (B)(ii), \( t = zx \) and so \( tb = zxb \). Thus
\[
[a, b] = [xa, xb].
\]
\( \square \)

**Lemma 3.9.** Let \( [a, b], [b, c] \in Q \). Then
\[
[a, b][b, c] = [a, c].
\]
Proof. We have 
\[ [a, b][b, c] = [xa, yc] \]
where \( xb = yb \) and \( x, y \in H_{r(x),r(b) - l(b) + \max(l(b),l(b))} \) so that \( x, y \in H_{r(x),r(b)} \). By (B)(i), \( x = y \). Then by Lemma 3.8 \([xa, xc] = [a, c] \). \( \square \)

Lemma 3.10. The semigroup \( Q \) is regular.

Proof. Let \([a, b] \in Q \). Then \([b, a] \in Q \) and by Lemma 3.9
\[ [a, b][b, a][a, b] = [a, b] \]
\( \square \)

Let \([a, a] \in Q \), then by Lemma 3.9 we have \([a, a][a, a] = [a, a] \), that is, \([a, a] \) is an idempotent in \( Q \). Hence \([\{a, a\}, a \in S \} \subseteq E(Q) \).

Lemma 3.11. The set of idempotents of \( Q \) is given by \( E(Q) = \{[a, a]; a \in S \} \).

Proof. Let \([a, b] \in E(Q) \), then \([a, b][a, b] = [a, b] \) and so \([xa, yb] = [a, b] \) where \( xb = ya \) for some \( x \in H_{r(x),r(b) - l(b) + \max(l(b),l(a))} \), \( y \in H_{r(x),r(a) - l(a) + \max(l(b),l(a))} \) so that
\[ xa \in H_{r(x),l(a) - l(b) + \max(l(b),l(a))} \] \( yb \in H_{r(x),l(b) - l(a) + \max(l(b),l(a))} \).

Since \([xa, yb] = [a, b] \), then there exist \( t, z \in S \) such that \( txa = za \) and \( tyb = zb \) where \( t \in H_{r(t),r(x)} \) and \( z \in H_{r(t),r(a)} \). It follows that \( l(tx) = l(a) \) and \( l(ty) = l(b) \). Hence
\[ tx \in H_{r(t),r(b) - l(b) + \max(l(b),l(a))} \] \( ty \in H_{r(t),r(a) - l(a) + \max(l(b),l(a))} \),
so that
\[ l(tx) \geq r(b) = r(a), l(ty) \geq r(a) = l(b) \] \( l(z) = r(a) = r(b) \).

By (B)(i), \( tx = z = ty \). From (B)(ii) as \( r(x) = r(y) = l(t) \), we have \( x = y \), and so \( l(x) = l(y) \), that is, \( r(a) - l(b) + \max(l(b),l(a)) = r(a) - l(a) + \max(l(b),l(a)) \).

Hence \( l(a) = l(b) \) which gives \( l(x) = r(a) = r(b) \). Since \( xb = ya = xa \) by (B)(ii) \( a = b \). \( \square \)

Lemma 3.12. The set \( E(Q) \) is w-chain.

Proof. Let \([a, a], [b, b] \in E(Q) \), then
\[ [a, a][b, b] = [xa, yb] \] where \( xa = yb \),
and \( x \in H_{r(x),r(a) - l(a) + \max(l(a),l(b))} \), \( y \in H_{r(x),r(b) - l(b) + \max(l(b),l(b))} \). Hence
\[ [a, a][b, b] = [xa, xa] = [yb, yb] \].

If \( l(a) \geq l(b) \), then \( x \in H_{r(x),r(a)} \) and so \( xa \in H_{r(x),l(a)} \). By Lemma 3.8 we have \([xa, xa] = [a, a] \). If \( l(b) \geq l(a) \), then \( y \in H_{r(x),r(b)} \) and \( yb \in H_{r(x),l(b)} \) so that \([yb, yb] = [b, b] \) by Lemma 3.8. \( \square \)
Notice also from Lemma 3.12 that if \( l(a) = l(b) \), then \([a,a][b,b] = [a,a] = [b,b] \).

By Lemma 3.12, the idempotents of \( Q \) form an \( \omega \)-chain and hence commute, by Lemma 3.10, the following Lemma is clear.

**Lemma 3.13.** The semigroup \( Q \) is inverse.

**Lemma 3.14.** The semigroup \( Q \) is a bisimple inverse semigroup.

**Proof.** To show that \( Q \) is a bisimple inverse semigroup, we need to prove that, for any two idempotents \([a,a], [b,b] \) in \( E(Q) \), there is \( q \in Q \) such that \( qq^{-1} = [a,a] \) and \( q^{-1}q = [b,b] \).

By (A), \( S\varphi \) is a left I-order in \( B \). By Lemma 2.2, \( S\varphi \) is straight, so that for \((l(a), l(b))\) there exist \( c, d \) in \( S \) such that

\[
(l(a), l(b)) = c\varphi^{-1}d\varphi \quad \text{where} \quad c\varphi R \ d\varphi \quad \text{in} \quad B,
\]

so that \( c\varphi = (u, l(a)) \) and \( d\varphi = (u, l(b)) \) for some \( u \in \mathbb{N}^0 \). Hence \( q = [c, d] \in Q \).

By Lemma 3.9, \( qq^{-1} = [c, d][d, c] = [c, c] \) and, similarly, \( q^{-1}q = [d, d] \). By the argument following Lemma 3.12, \( [c, c] = [a, a] \) and \([d, d] = [b, b] \), as required. \( \square \)

The following lemma throws full light on the relationship between \( S \) and \( Q \).

**Lemma 3.15.** Every element of \( Q \) can be written as \((a\theta)^{-1}b\theta\), where \( a, b \in S \).

**Proof.** Suppose that \( q = [a,b] \in Q \). In view of Lemma 3.7, \( a\theta = [x, xa] \) and \( b\theta = [y, yb] \) respectively, for some \( x \in H_{r(x), r(a)} \) and \( y \in H_{r(y), r(b)} \). Hence

\[
(a\theta)^{-1}b\theta = [xa, x][y, yb] = [txa, hyb] \quad \text{where} \quad tx = hy, r(t) = r(h), l(t) = r(x) \text{ and } l(h) = r(y) = [txa, txb] \quad \text{where} \quad l(tx) = r(a) = [a, b] \quad \text{by Lemma 3.8}
\]

\( \square \)

From Lemmas 3.7, 3.12, 3.13, 3.14 and 3.15, we deduce that \( S \) is a straight left I-order in a bisimple inverse \( \omega \)-semigroup. \( \square \)

**References**


Department of Mathematics, University of York, Heslington, York YO10 5DD, UK

E-mail address: ng521@york.ac.uk
No. 1  'G-COMPLETE REDUCIBILITY AND SEMISIMPLE MODULES'  Michael Bate, Sebastian Herpel, Benjamin Martin & Gerhard Röhre

No. 2  'RIGHT-ANGLED COXETER POLYTOPES, HYPERBOLIC 6-MANIFOLDS, AND A PROBLEM OF SIEGEL'  Brent Everitt, John Ratcliffe & Steven Tschantz

No. 3  'AFFINE CONSTELLATIONS WITHOUT MUTUALLY UNBIASED COUNTERPARTS'  Stefan Weigert & Thomas Durt

No. 4  'THE EVERETT-WHEELER INTERPRETATION AND THE OPEN FUTURE'  Anthony Sudbery

No. 5  'CENTRAL LIMIT THEOREM FOR ASSOCIATED CLASS FUNCTIONS ON THE SYMMETRIC GROUP'  Dirk Zeindler

No. 6  'SINGULARITY THEOREMS FROM WEAKENED ENERGY CONDITIONS'  Chris Fewster & Gregory Galloway

No. 7  'PROPER RESTRICTION SEMIGROUPS AND PARTIAL ACTIONS'  Claire Cornock & Victoria Gould

No. 8  'AXIOMATISABILITY PROBLEMS FOR S-POSETS'  Victoria Gould & Lubna Shaheen

No. 9  'LEFT ADEQUATE AND LEFT EHRESMANN MONOIDS'  Victoria Gould, Mario Branco & Gracinda Gomes

No. 10  'SEMIGROUPS OF INVERSE QUOTIENTS'  Nassraddin Ghroda & Victoria Gould

No. 11  'PRIMITIVE INVERSE SEMIGROUPS OF LEFT I-QUOTIENTS'  Nassraddin Ghroda

No. 12  'BISIMPLE INVERSE ω-SEMIGROUPS OF LEFT I-QUOTIENTS'  Nassraddin Ghroda