TITLE:
‘INJECTIVE SCHUR MODULES’

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Injective Schur Modules

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Abstract

We determine for which partitions $\lambda$ the corresponding induced module (or Schur module in the language of Buchsbaum et. al., [1]) $\nabla(\lambda)$ is injective in the category of polynomial modules for a general linear group over an infinite field. Since the problem is essentially no more difficult in the quantised case we address it at this level of generality.

Introduction

Let $K$ be field and $0 \neq q \in K$. Let $G(n)$ be the corresponding quantum general linear of degree $n$, as in, for example [7]. Let $T(n)$ denote the algebraic torus and $B(n)$ the Borel subgroup, as in [7]. Each partition $\lambda$ with at most $n$ parts determines a one dimensional $T(n)$-module $K_\lambda$ and the module structure extends uniquely to give the structure of a $B(n)$-module. We give a combinatorial description of those $\lambda$ such that the induced module $\nabla(\lambda) = \text{ind}_{B(n)}^{G(n)} K_\lambda$ is injective as a polynomial module. Section 1 is preliminary and the description is given in Section 2.

1 Preliminaries

1.1 Combinatorics

The standard reference for the polynomial representation theory of $GL_n(K)$ is the monograph [9]. Though we work in the quantised context this reference is appropriate as the combinatorial set-up is essentially the same and we adopt the notation of [9] wherever convenient. Further details may also be found in the monograph, [7], which treats the quantised case.
By a partition we mean an infinite sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers with $\lambda_1 \geq \lambda_2 \geq \ldots$ and $\lambda_j = 0$ for all $j$ sufficiently large. If $m$ is a positive integer such that $\lambda_j = 0$ for $j > m$ we identify $\lambda$ with the finite sequence $(\lambda_1, \ldots, \lambda_m)$. The length $\text{len}(\lambda)$ of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is 0 if $\lambda = 0$ and is the positive integer $m$ such that $\lambda_m \neq 0$, $\lambda_{m+1} = 0$, if $\lambda \neq 0$.

For a partition $\lambda$, we denote by $\lambda'$ the transpose partition of $\lambda$. We define the degree of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ by $\deg(\lambda) = \lambda_1 + \lambda_2 + \cdots$.

We set $X(n) = \mathbb{Z}^n$. There is a natural partial order on $X(n)$. For $\lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n) \in X(n)$, we write $\lambda \leq \mu$ if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for $i = 1, 2, \ldots, n - 1$ and $\lambda_1 + \cdots + \lambda_n = \mu_1 + \cdots + \mu_n$. We shall use the standard $\mathbb{Z}$-basis $\epsilon_1, \ldots, \epsilon_n$ of $X(n)$. Thus $\epsilon_i = (0, \ldots, 1, \ldots, 0)$ (with 1 in the $i$th position), for $1 \leq i \leq n$. We write $\omega_i$ for the element $\epsilon_1 + \cdots + \epsilon_i$ of $X(n)$, for $1 \leq i \leq n$, and set $\omega_0 = 0$. We write $\Lambda(n)$ for the set of $n$-tuples of nonnegative integers. We write $X^+(n)$ for the set of dominant $n$-tuples of integers, i.e., the set of elements $\lambda = (\lambda_1, \ldots, \lambda_n) \in X(n)$ such that $\lambda_1 \geq \cdots \geq \lambda_n$.

We write $\Lambda^+(n)$ for the set of partitions into at most $n$-parts, i.e., $\Lambda^+(n) = X^+(n) \cap \Lambda(n)$. For a nonnegative integer $r$ we write $\Lambda^+(n, r)$ for the set of partitions of $r$ into at most $n$ parts, i.e., the set of elements of $\Lambda^+(n)$ of degree $r$.

### 1.2 Rational and Polynomial Modules

Appropriate references for the set-up described here are [3], [6], [7]. Let $K$ be a field. If $V, W$ are vector spaces over $K$, we write $V \otimes W$ for the tensor product $V \otimes_K W$. We shall be working with the representation theory of quantum groups over $K$. By the category of quantum groups over $K$ we understand the opposite category of the category of Hopf algebras over $K$.

Less formally we shall use the expression “$G$ is a quantum group” to indicate that we have in mind a Hopf algebra over $K$ which we denote $K[G]$ and call the coordinate algebra of $G$. We say that $\phi : G \rightarrow H$ is a morphism of quantum groups over $K$ to indicate that we have in mind a morphism of Hopf algebras over $K$, from $K[H]$ to $K[G]$, denoted $\phi^\sharp$ and called the co-morphism of $\phi$. We will say $H$ is a quantum subgroup of the quantum group $G$, over $K$, to indicate that $H$ is a quantum group with coordinate algebra $K[H] = K[G]/I$, for some Hopf ideal $I$ of $K[G]$, which we call the defining ideal of $H$. The inclusion morphism $i : H \rightarrow G$ is the morphism of quantum groups whose co-morphism $i^\sharp : K[G] \rightarrow K[H] = K[G]/I$ is the natural map.

Let $G$ be a quantum group over $K$. The category of left (resp. right) $G$-modules is the the category of right (resp. left) $K[G]$-comodules. We write $\text{Mod}(G)$ for the category of left $G$-modules and $\text{mod}(G)$ for the category of finite dimensional left $G$-modules. We shall also call a $G$-module a rational $G$-module (by analogy with the representation theory of algebraic groups). A $G$-module will mean a left $G$-module unless indicated otherwise. For a
finite dimensional $G$-module $V$ the dual space $V^* = \text{Hom}_K(V,K)$ has a natural $G$-module structure. For a finite dimensional $G$-module $V$ and a non-negative integer $r$ we write $V^{\otimes r}$ for the $r$-fold tensor product $V \otimes V \otimes \cdots \otimes V$ and write $V^{\otimes -r}$ for the dual of $V^{\otimes r}$.

Let $V$ be a finite dimensional $G$-module with structure map $\tau : V \to V \otimes K[G]$. The coefficient space $\text{cf}(V)$ of $V$ is the subspace of $K[G]$ spanned by the “coefficient elements” $f_{ij}$, $1 \leq i, j \leq m$, defined with respect to a basis $v_1, \ldots, v_m$ of $V$, by the equations

$$\tau(v_i) = \sum_{j=1}^{m} v_j \otimes f_{ji}$$

for $1 \leq i \leq m$. The coefficient space $\text{cf}(V)$ is independent of the choice of basis and is a subcoalgebra of $K[G]$.

We fix $0 \neq q \in K$. For a positive integer $n$ we shall be working with the corresponding quantum general linear group $G(n)$, as in [7]. We have a $K$-bialgebra $A(n)$ given by generators $c_{ij}$, $1 \leq i, j \leq n$, subject to certain relations (depending on $q$), as in [7], 0.20. The comultiplication map $\delta : A(n) \to A(n) \otimes A(n)$ satisfies $\delta(c_{ij}) = \sum_{r=1}^{n} c_{ir} \otimes c_{rj}$ and the augmentation map $\epsilon : A(n) \to K$ satisfies $\epsilon(c_{ij}) = \delta_{ij}$ (the Kronecker delta), for $1 \leq i, j \leq n$. The elements $c_{ij}$ will be called the coordinate elements and we define the determinant element

$$d_n = \sum_{\pi \in \text{Sym}(n)} \text{sgn}(\pi)c_{1,\pi(1)} \cdots c_{n,\pi(n)}.$$

Here $\text{sgn}(\pi)$ denotes the sign of the permutation $\pi$. We form the Ore localisation $A(n)_{d_n}$. The comultiplication map $A(n) \to A(n) \otimes A(n)$ and augmentation map $A(n) \to K$ extend uniquely to $K$-algebraic maps $A(n)_{d_n} \to A(n)_{d_n} \otimes A(n)_{d_n}$ and $A(n)_{d_n} \to K$, giving $A(n)_{d_n}$ the structure of a Hopf algebra. By the quantum general linear group $G(n)$ we mean the quantum group over $K$ with coordinate algebra $K[G(n)] = A(n)_{d_n}$.

We write $T(n)$ for the quantum subgroup of $G(n)$ with defining ideal generated by all $c_{ij}$ with $1 \leq i, j \leq n$, $i \neq j$. We write $B(n)$ for quantum subgroup of $G(n)$ with defining ideal generated by all $c_{ij}$ with $1 \leq i < j \leq n$. We call $T(n)$ a maximal torus and $B(n)$ a Borel subgroup of $G(n)$ (by analogy with the classical case).

We now recall the weight space decomposition of a finite dimensional $T(n)$-module. For $1 \leq i \leq n$ we define $\tilde{e}_{ii} = c_{ii} + I_{T(n)} \in K[T(n)]$, where $I_{T(n)}$ is the defining ideal of the quantum subgroup $T(n)$ of $G(n)$. Note that $\tilde{e}_{11} \cdots \tilde{e}_{nn} = d_n + I_{T(n)}$, in particular each $\tilde{e}_{ii}$ is invertible in $K[T(n)]$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in X(n)$ we define $\tilde{e}^\lambda = \tilde{e}_{11}^{\lambda_1} \cdots \tilde{e}_{nn}^{\lambda_n}$. The elements $\tilde{e}^\lambda$, $\lambda \in X(n)$, are group-like and form a $K$-basis of $K[T(n)]$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in X(n)$, we write $K_\lambda$ for $K$ regarded as a (one dimensional) $T(n)$-module with structure map $\tau : K_\lambda \to K_\lambda \otimes K[T(n)]$ given by $\tau(v) =$
$v \otimes e^\lambda$, $v \in K_\lambda$. For a finite dimensional rational $T(n)$-module $V$ with structure map $\tau : V \to V \otimes K[T(n)]$ and $\lambda \in X(n)$ we have the weight space

$$V^\lambda = \{ v \in V | \tau(v) = v \otimes e^\lambda \}.$$ 

Moreover, we have the weight space decomposition $V = \bigoplus_{\lambda \in X(n)} V^\lambda$. We say that $\lambda \in X(n)$ is a weight of $V$ if $V^\lambda \neq 0$.

For each $\lambda \in X^+(n)$ there is an irreducible rational $G(n)$-module $L_n(\lambda)$ which has unique highest weight $\lambda$ and such $\lambda$ occurs as a weight with multiplicity one. The modules $L_n(\lambda)$, $\lambda \in X^+(n)$, form a complete set of pairwise non-isomorphic irreducible rational $G(n)$-modules. Note that for $\lambda = (\lambda_1, \ldots, \lambda_n) \in X^+(n)$ the dual module $L_n(\lambda)^*$ is isomorphic to $L_n(\lambda^*)$, where $\lambda^* = (-\lambda_n, \ldots, -\lambda_1)$. For a finite dimensional rational $G(n)$-module $V$ and $\lambda \in X^+(n)$ we write $[V : L_n(\lambda)]$ for the multiplicity of $L_n(\lambda)$ as a composition factor of $V$.

We write $D_n$ for the one dimensional $G(n)$-module corresponding to the determinant. Thus $D_n$ has structure map $\tau : D_n \to D_n \otimes K[G]$, given by $\tau(v) = v \otimes d_n$, for $v \in D_n$. Thus we have $D_n = L_n(\omega) = L_n(1, 1, \ldots, 1)$. We write $E_n$ for the natural $G(n)$-module. Thus $E_n$ has basis $e_1, \ldots, e_n$, and the structure map $\tau : E_n \to E_n \otimes K[G(n)]$ is given by $\tau(e_i) = \sum_{j=1}^n e_j \otimes c_{ij}$.

A finite dimensional $G(n)$-module $V$ is called polynomial if $\text{cf}(V) \leq A(n)$. The modules $L_n(\lambda)$, $\lambda \in \Lambda^+(n)$, form a complete set of pairwise non-isomorphic irreducible polynomial $G(n)$-modules. We write $I_n(\lambda)$ for the injective envelope of $L_n(\lambda)$ in the category of polynomial modules. We have a grading $A(n) = \bigoplus_{r=0}^\infty A(n, r)$ in such a way that each $c_{ij}$ has degree 1. Moreover each $A(n, r)$ is a finite dimensional subcoalgebra of $A(n)$. The dual algebra $S(n, r)$ is known as the $q$-Schur algebra. A finite dimensional $G(n)$-module $V$ is polynomial of degree $r$ if $\text{cf}(V) \leq A(n, r)$. We write $\text{pol}(n)$ (resp. $\text{pol}(n, r)$) for the full subcategory of $\text{mod}(G(n))$ whose objects are the polynomial modules (resp. the modules which are polynomial of degree $r$).

For an arbitrary finite dimensional polynomial $G(n)$-module we may write $V$ uniquely as a direct sum $V = \bigoplus_{r=0}^\infty V(r)$ in such a way that $V(r)$ is polynomial of degree $r$, for $r \geq 0$. Let $r \geq 0$. The modules $L_n(\lambda)$, $\lambda \in \Lambda^+(n, r)$, form a complete set of pairwise non-isomorphic irreducible polynomial $G(n)$-modules which are polynomial of degree $r$. We write $\text{mod}(S)$ for the category of left modules for a finite dimensional $K$-algebra $S$. The category $\text{pol}(n, r)$ is naturally equivalent to the category $\text{mod}(S(n, r))$. It follows in particular that, for $\lambda \in \Lambda^+(n, r)$, the module $I_n(\lambda)$ is a finite dimensional module which is polynomial of degree $r$.

We shall also need modules induced from $B(n)$ to $G(n)$. (For details of the induction functor $\text{Mod}(B(n)) \to \text{Mod}(G(n))$ see, for example, [6].) For $\lambda \in X(n)$ there is a unique (up to isomorphism) one dimensional $B(n)$-module whose restriction to $T(n)$ is $K_\lambda$. We also denote this module by $K_\lambda$. The induced module $\text{ind}_{B(n)}^{G(n)} K_\lambda$ is non-zero if and only if $\lambda \in X^+(n)$. For
\( \lambda \in X^+(n) \) we set \( \nabla_n(\lambda) = \text{ind}_{B(n)}^G K_\lambda \). Then \( \nabla_n(\lambda) \) is finite dimensional (and its character is the Schur symmetric function corresponding to \( \lambda \)). The \( G(n) \)-module socle of \( \nabla_n(\lambda) \) is \( L_n(\lambda) \). The module \( \nabla_n(\lambda) \) has unique highest weight \( \lambda \) and this weight occurs with multiplicity one.

A filtration \( 0 = V_0 \leq V_1 \leq \cdots \leq V_r = V \) of a finite dimensional rational \( G(n) \)-module \( V \) is said to be good if for each \( 1 \leq i \leq r \) the quotient \( V_i/V_{i-1} \) is either zero or isomorphic to \( \nabla_n(\lambda^i) \) for some \( \lambda^i \in X^+(n) \). For a rational \( G(n) \)-module \( V \) admitting a good filtration for each \( \lambda \in X^+(n) \), the multiplicity \( |\{1 \leq i \leq r \mid V_i/V_{i-1} \cong \nabla_n(\lambda)\}| \) is independent of the choice of the good filtration, and will be denoted \( (V : \nabla_n(\lambda)) \).

For a partition \( \lambda \) we denote by \([\lambda]\) the corresponding partition diagram (as in [9]). For a positive integer \( l \), the \( l \)-core of \([\lambda]\) is the diagram obtained by removing skew \( l \)-hooks, as in [10]. If \( \lambda, \mu \in \Lambda^+(n,r) \) and \([\lambda]\) and \([\mu]\) have different \( l \)-cores then the simple modules \( L_n(\lambda) \) and \( L_n(\mu) \) belong to different blocks and it follows in particular that \( \text{Ext}^i_{S(n,r)}(\nabla(\lambda), \nabla(\mu)) = 0 \), for all \( i \geq 0 \). A precise description of the blocks of the \( q \)-Schur algebras was found by Cox, see [2], Theorem 5.3.

For \( \lambda \in \Lambda^+(n) \) the module \( I_n(\lambda) \) has a good filtration and we have the reciprocity formula \( (I_n(\lambda) : \nabla_n(\mu)) = [\nabla_n(\mu) : L_n(\lambda)] \) see e.g., [6], Section 4, (6).

\section{Injective Partitions.}

We are interested in giving a combinatorial description of those \( \lambda \in \Lambda^+(n) \) such that the induced module \( \nabla_n(\lambda) \) is injective in the category of polynomial \( G(n) \)-modules. However, if \( q \) is not a root of unity or if \( q = 1 \) and \( K \) has characteristic 0 then all \( G(n) \)-modules are semisimple, see e.g., [3], (3.3.2) or [6], Section 4, (8). We assume from now on that \( q \) is a root of unity and that if \( q = 1 \) then \( K \) has positive characteristic.

Let \( l \) be the smallest positive integer such that \( 1 + q + \cdots + q^{l-1} = 0 \). Thus \( l \) is the order of \( q \) if \( q \neq 1 \) and \( l \) is the characteristic of \( K \) if \( q = 1 \) and \( K \) has positive characteristic.

We write \( X_1(n) \) for the set of \( l \)-restricted partition into at most \( n \) parts, i.e., the set of elements \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+(n) \) such that \( 0 \leq \lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, \lambda_n < l \).

Let \( \lambda \in \Lambda^+(n) \). Recall that the induced module \( \nabla_n(\lambda) \) has simple socle \( L_n(\lambda) \), so that \( \nabla_n(\lambda) \) embeds in \( I_n(\lambda) \). We are interested in the cases in which this embedding is an isomorphism.

\textbf{Definition 2.1.} We call an element \( \lambda \) of \( \Lambda^+(n) \) an injective partition for \( G(n) \), or just an injective partition relative to \( n \), if \( \nabla_n(\lambda) \) is injective in the category of polynomial \( G(n) \)-modules, i.e., if \( \nabla_n(\lambda) = I_n(\lambda) \).
Let $\lambda, \mu \in \Lambda^+(n, r)$. We may also consider $\lambda$ and $\mu$ as elements of $\Lambda^+(N)$ for $N \geq n$ and we have $[\nabla_n(\lambda) : L_n(\mu)] = [\nabla_N(\lambda) : L_N(\mu)]$, by [7], 4.2, (6) (see [9], (6.6e) Theorem for the classical case). We shall write simply $[\lambda : \mu]$ for $[\nabla_n(\lambda) : L_n(\mu)]$.

**Remark 2.2.** Let $\lambda \in \Lambda^+(n)$ and suppose $\lambda$ has degree $r$. For $\mu \in \Lambda^+(n, r)$ we have $(I_n(\lambda) : \nabla_n(\mu)) = [\mu : \lambda]$. In particular we have $(I_n(\lambda) : \nabla_n(\lambda)) = 1$ and if $(I_n(\lambda) : \nabla_n(\mu)) \neq 0$ then $\mu \geq \lambda$. Thus $\lambda$ is injective for $G(n)$ if and only if $[\mu : \lambda] = 0$ for all $\mu \in \Lambda^+(n, r)$ with $\mu > \lambda$.

Suppose $\lambda$ is injective for $G(n)$ and $N \geq n$. Let $\mu \in \Lambda^+(N, r)$ and suppose $\mu > \lambda$. Then $\mu$ has at most $n$ parts, i.e., $\mu \in \Lambda^+(n, r)$, and therefore $[\mu : \lambda] = 0$. Thus if $\lambda$ is injective for $G(n)$ then it is injective for $G(N)$ for all $N \geq n$.

From now on we shall simply say that a partition $\lambda$ is injective if it is injective for some, and hence every, $G(n)$ with $n \geq \text{len}(\lambda)$.

Henceforth, for a partition $\lambda$, we write simply $\nabla(\lambda)$ for $\nabla_n(\lambda)$, write $L(\lambda)$ for $L_n(\lambda)$ and so on, with $n$ understood to be sufficiently large, where confusion seems unlikely.

**Lemma 2.3.** If $\lambda$ is injective and $n = \text{len}(\lambda)$ then $\lambda - \omega_n$ is injective.

**Proof.** We work with $G(n)$-modules. Suppose that $\mu$ is a partition bigger than $\lambda - \omega_n$ in the dominance order. We have

$$[\nabla_n(\mu) : L_n(\lambda - \omega_n)] = [D_n \otimes \nabla_n(\mu) : D_n \otimes L_n(\lambda - \omega_n)] = [\mu + \omega_n : \lambda]$$

and this is 0 since $\mu + \omega_n > \lambda$. Hence $\lambda - \omega_n$ is injective by Remark 2.2.

**Lemma 2.4.** A partition $\lambda$ is injective if and only if $\lambda$ is a maximal weight of $I(\lambda)$.

**Proof.** The module $\nabla(\lambda)$ has maximal weight $\lambda$ so if $\lambda$ is injective it is a maximal weight of $I(\lambda)$.

Suppose conversely that $\lambda$ is a maximal weight of $I(\lambda)$. Let $\mu \in \Lambda^+(n, r)$ with $(I(\lambda) : \nabla(\mu)) \neq 0$ and hence, by reciprocity, $[\mu : \lambda] \neq 0$. Then $\mu \geq \lambda$ and by maximality $\mu = \lambda$ and so $\lambda$ is injective, by Remark 2.2.

Given a partition $\lambda$ we may write $\lambda$ uniquely in the form $\lambda = \lambda^0 + l \bar{\lambda}$, where $\lambda^0, \bar{\lambda}$ are partitions and $\lambda^0$ is $l$-restricted.

It will be important for us to make a comparison with the classical case $q = 1$. In this case we will write $\hat{G}(n)$ for $G(n)$ and write $x_{ij}$ for the coordinate element $c_{ij}$, $1 \leq i, j \leq n$. In this case we write $\hat{L}_n(\lambda)$ for the $\hat{G}(n)$-module $L_n(\lambda)$, $\lambda \in X^+(n)$.

Now we have a morphism of Hopf algebras $\theta : K[\hat{G}(n)] \to K[G(n)]$ given by $\theta(x_{ij}) = c_{ij}'$, for $1 \leq i, j \leq n$. We write $F : G(n) \to G(n)$ for the
morphism of quantum groups such that $F^\sharp = \theta$. Given a $\hat{G}(n)$-module $V$ we write $V^F$ for the corresponding $G(n)$-module. Thus, $V^F$ as a vector space is $V$ and if the $\hat{G}(n)$-module $V$ has structure map $\tau : V \to V \otimes K[\hat{G}(n)]$ then $V^F$ has structure map $(\text{id}_V \otimes F) \circ \tau : V^F \to V^F \otimes K[G(n)]$, where $\text{id}_V : V \to V$ is the identity map on the vector space $V$.

We have the following relationship between the irreducible modules for $G(n)$ and $\hat{G}(n)$, see [7], Section 3.2, (5).

**Theorem 2.5. (Steinberg’s Tensor Product Theorem)** For $\lambda^0 \in X_1(n)$ and $\tilde{\lambda} \in X^+(n)$ we have

$$L_n(\lambda^0 + l\tilde{\lambda}) \cong L_n(\lambda^0) \otimes \hat{L}_n(\tilde{\lambda})^F.$$

**Lemma 2.6.** If $\lambda$ is an injective partition for $G(n)$ then $\lambda^0$ is injective for $G(n)$ and $\tilde{\lambda}$ is injective for $\hat{G}(n)$.

**Proof.** We write $G_1$ for the first infinitesimal subgroup of $G(n)$. The $G_1$-socle of $\nabla(\lambda)$ is $L(\lambda^0) \otimes \nabla(\tilde{\lambda})^F$, and the $G_1$-socle of $I(\lambda)$ is $L(\lambda^0) \otimes I(\tilde{\lambda})^F$ by [8], Lemma 3.2 (i) (and the remarks on the quantised situation in [8], Section 5). Since $\nabla(\tilde{\lambda})^F$ embeds in $I(\tilde{\lambda})^F$ we must have $\nabla(\lambda) = I(\lambda)$ and $\tilde{\lambda}$ is injective for $\hat{G}(n)$.

Let $\mu$ be a maximal weight of $I(\lambda^0)$. Now by [8], Lemma 3.1, $I(\lambda^0) \otimes I(\tilde{\lambda})^F$ has $G(n)$-socle $L(\lambda)$ and so $I(\lambda^0) \otimes I(\tilde{\lambda})^F$ embeds in $I(\lambda)$. Thus $\mu + l\tilde{\lambda}$ is a weight of $I(\lambda)$ and so $I(\lambda)$ has a maximal weight $\tau$, say, such that $\tau \geq \mu + l\tilde{\lambda}$.

But $I(\lambda) = \nabla(\lambda)$ has unique maximal weight $\lambda$ so that $\lambda \geq \tau \geq \mu + l\tilde{\lambda} \geq \lambda^0 + l\tilde{\lambda} = \lambda$ and so $\mu = \lambda^0$. Hence $\lambda^0$ is a maximal weight of $I(\lambda^0)$ and so, by Lemma 2.4, $\lambda^0$ is injective.

**Lemma 2.7.** Let $\lambda$ be an injective partition and write $\lambda = \lambda^0 + l\tilde{\lambda}$, for partitions $\lambda^0, \tilde{\lambda}$ with $\lambda^0$ being $l$-restricted. Then $\lambda^0$ is an $l$-core.

**Proof.** By the previous lemma we may assume $\lambda = \lambda^0$, i.e., that $\lambda$ is restricted. Thus $I(\lambda)$ is isomorphic to its contravariant dual, see e.g., [7], 4.3,(2),(ii) , 4.3, (4) and (4.3), (ix). Hence $I(\lambda)$ has simple head $L(\lambda)$. But $I(\lambda) = \nabla(\lambda)$ and $[\nabla(\lambda) : L(\lambda)] = 1$ so that in fact $I(\lambda) = \nabla(\lambda) = L(\lambda)$. Thus we get $[\mu : \lambda] = \delta_{\lambda,\mu}$ (the Kronecker delta) and $[\lambda : \tau] = \delta_{\lambda,\tau}$, for all partitions $\mu, \tau$ with $|\mu| = |\tau| = |\lambda|$. Hence $L(\lambda)$ is the only simple in its block (up to isomorphism), i.e., $\lambda$ is an $l$-core.

We introduce some additional notation. We set $\delta_0 = 0$ and $\delta_n = (n,n - 1,\ldots,2,1)$, for $n \geq 1$. We set $\sigma_0 = 0$ and

$$\sigma_n = (n(l-1),(n-1)(l-1),\ldots,2(l-1),(l-1))$$

for $n \geq 1$, so that $\sigma_n = (l-1)\delta_n$, $n \geq 0$.  

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We call the partitions of the form $\sigma_n$, for some $n \geq 0$, the Steinberg partitions. The justification for this is that in the classical case, with $K$ an algebraically closed field of characteristic $p > 0$ the restriction of the $GL_{n+1}(K)$-module $L(\sigma_n)$ to the special linear group $SL_{n+1}(K)$ is the usual Steinberg module.

Note that, since

\[(n(l-1)+1, \ldots, 2(l-1)+1, l) = (l, \ldots, l) + ((n-1)(l-1), \ldots, l-1, 0)\]

we have $\sigma_n + \omega_n = \sigma_{n-1} + l\omega_n$ i.e.,

\[\sigma_n = \sigma_{n-1} + (l-1)\omega_n\]

for $n \geq 1$.

**Remark 2.8.** Suppose $n \geq 1$, $0 \leq a < l$ and let $\mu$ be an injective partition of length at most $n$. We note that $\lambda = \sigma_{n-1} + a\omega_n + l\mu$ is injective. We have that $\nabla(\sigma_{n-1} + a\omega_n) = \nabla(\sigma_{n-1}) \otimes D_n^{a\lambda}$ is injective as a module for the first infinitesimal subgroup $G_1$ of $G(n)$ by [7], Section 3.2, (12) (and for example [11], II, 10.2 Proposition in the classical case). Hence by [8], Lemma 3.2(ii), and the remarks on the quantised situation in [8], Section 5, we have $I(\sigma_{n-1} + a\omega_n) = \nabla(\sigma_{n-1} + a\omega_n)$ and $I(\lambda) = \nabla(\sigma_{n-1} + a\omega_n) \otimes I(\mu)^F = \nabla(\sigma_{n-1} + a\omega_n) \otimes \nabla(\mu)^F$. However, by [7], Section 3.2, (13) (and [11], II, 3.19 Proposition in the classical case) we have $\nabla(\lambda) = \nabla(\sigma_{n-1} + a\omega_n) \otimes \nabla(\mu)^F$ so that $\lambda = \sigma_{n-1} + a\omega_n + l\mu$ is injective.

**Remark 2.9.** Suppose $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+(n)$ is an $l$-core and $\lambda_n = l-1$. Then we have $\lambda = \sigma_n$. No doubt this is well known. We see it as follows. We may assume $n \geq 2$. Certainly $\lambda_{n-1} - \lambda_n < l$, for otherwise row $n-1$ of the diagram of $\lambda$ contains a skew $l$-hook. If $\lambda_{n-1} < 2l - 2$ then there is a skew $l$-hook beginning at $(n-1, \lambda_{n-1})$ and ending at $(n, \lambda_{n-1} + 2l - l)$. Thus we have $\lambda_{n-1} = 2l - 2$. Now $\mu = \lambda - (l-1)\omega_n$ is a $l$-core of length $n-1$ with last non-zero entry $l-1$. Hence we can assume inductively that $\mu = \sigma_{n-1}$ and hence

\[\lambda = \sigma_{n-1} + (l-1)\omega_n = \sigma_n.\]

**Lemma 2.10.** If the partition $\lambda$ is injective and $\text{len}(\lambda^0) < \text{len}(\bar{\lambda})$ then $\lambda^0 = \sigma_{n-1}$, where $n = \text{len}(\lambda)$

*Proof.* We consider $\mu = \lambda - \omega_n$. Note that $\mu$ has length $n$ and $\mu_n$ is congruent to $-1$ modulo $l$. Hence, writing $\mu = \mu^0 + l\bar{\mu}$, we have $\mu_n^0 = l-1$. Moreover, $\mu$ is injective, by Lemma 2.3, and so $\mu^0$ is injective by Lemma 2.6. Hence $\mu^0$ is a core, by Lemma 2.7 and $\mu^0 = \sigma_n$, by Remark 2.9. Now we have

\[\lambda = \mu + \omega_n = \sigma_n + \omega_n + l\bar{\mu} = \sigma_{n-1} + l(\bar{\mu} + \omega_n)\]

and so $\lambda^0 = \sigma_{n-1}$. 

\[\square\]
Lemma 2.11. Let $\lambda$ be a partition of length $n$. If $\lambda$ is injective then $\text{len}(\bar{\lambda}) \leq \text{len}(\lambda^0) + 1$ and in case equality holds we have $\lambda^0 = \sigma_{n-1}$.

Proof. If $\text{len}(\bar{\lambda}) \geq \text{len}(\lambda^0) + 1$ then $\text{len}(\bar{\lambda}) > \text{len}(\lambda^0)$ so that $n = \text{len}(\bar{\lambda})$ and $\text{len}(\lambda^0) < n$. Hence $\lambda^0 = \sigma_{n-1}$ by Lemma 2.10 and $\text{len}(\bar{\lambda}) = \text{len}(\lambda^0) + 1$.

Lemma 2.12. Suppose that the partition $\lambda$ satisfies $\text{len}(\lambda^0) = \text{len}(\lambda)$ and $\lambda^0$ is an $l$-core. If $\lambda - \omega_n$ is injective, where $n$ is the length of $\lambda$, then so is $\lambda$.

Proof. Suppose $\mu$ is a partition such that $\mu > \lambda$ and $[\mu : \lambda] \neq 0$. Then $\mu$ also has core $\lambda^0$ and so $\mu$ has length $n$. Thus we may write $\mu = \tau + \omega_n$, for some partition $\tau$. But then

$$[\mu : \lambda] = [\tau + \omega_n : \lambda] = [\tau : \lambda - \omega_n] = 0.$$ 

Thus no such partition $\mu$ exists and $\lambda$ is injective.

Definition 2.13. We define the Steinberg index $\text{stind}_l(\lambda)$ relative to $l$ of a partition $\lambda$ to be 0 if $\lambda_1 - \lambda_2 \neq l - 1$ and otherwise to be $m > 0$ if $\lambda_i - \lambda_{i+1} = l - 1$ for $1 \leq i \leq m$ and $\lambda_{m+1} - \lambda_{m+2} \neq l - 1$. (Thus for example $\text{stind}(\sigma_n) = n$, for $n \geq 0$.)

Proposition 2.14. Let $\lambda$ be a partition written $\lambda = \lambda^0 + l\bar{\lambda}$ in standard form. Then $\lambda$ is injective if and only if $\lambda^0$ is an $l$-core, $\bar{\lambda}$ is injective and $\text{len}(\lambda) \leq \text{stind}_l(\lambda^0) + 1$.

Proof. Let $n = \text{len}(\lambda)$.

We first suppose $\lambda$ is injective. Then $\bar{\lambda}$ is injective, by Lemma 2.6 and $\lambda^0$ is an $l$-core, by Lemma 2.7. We claim that also $\text{len}(\bar{\lambda}) \leq \text{stind}_l(\lambda^0) + 1$.

We know that $\text{len}(\bar{\lambda}) \leq \text{len}(\lambda^0) + 1$, by Lemma 2.11. Moreover, if $\text{len}(\bar{\lambda}) = \text{len}(\lambda^0) + 1$ then $\lambda^0 = \sigma_{n-1}$ and so $\text{stind}_l(\lambda^0) = n - 1$, $\text{len}(\bar{\lambda}) = n$, by Lemma 2.11, and the desired conclusion holds. Now suppose that the claim is false and that $\lambda$ is an injective partition of minimal degree for which it fails. Thus we have $\text{len}(\bar{\lambda}) \leq \text{len}(\lambda^0) = n$ by the case already considered. Thus we must have that $n \geq 2$ and that $\text{stind}_l(\lambda^0) = m$, say, is at most $n - 2$. Now $\mu = \lambda - \omega_n = (\lambda^0 - \omega_n) + p\bar{\lambda}$ is injective, by Lemma 2.3. But we have $\text{stind}_l(\lambda^0 - \omega_n) = \text{stind}_l(\lambda^0)$ and so, by minimality, $\text{len}(\bar{\lambda}) \leq \text{stind}_l(\lambda^0 - \omega_n) = \text{stind}_l(\lambda^0)$ and the claim is proved.

We now suppose that $\lambda$ is injective, that $\lambda^0$ is an $l$-core and $\text{len}(\bar{\lambda}) \leq \text{stind}_l(\lambda^0) + 1$. We show that $\lambda$ is injective by induction on the degree of $\lambda$.

If the Steinberg index of $\lambda$ is $n$ then $\lambda^0 = \sigma_n$ and $\lambda$ is injective by Remark 2.8.

If the Steinberg index of $\lambda$ is $n - 1$ then $\lambda^0$ has the form $\sigma_{n-1} + a\omega_n$, for some $0 \leq a < l$ and this case is also covered by Remark 2.8.
Thus we may assume that \( \text{stind}_l(\lambda) < n - 1 \). Then \( \text{len}(\lambda) < n \) so that \( \text{len}(\lambda^0) = n \). By Lemma 2.12 it is enough to show that \( \lambda - \omega_n \) is injective. But we have

\[
\lambda - \omega_n = (\lambda^0 - \omega_n) + p\bar{\lambda}
\]

and so \( \text{stind}_l(\lambda^0 - \omega_n) = \text{stind}_l(\lambda^0) \) and we are done by induction.

This solves the problem of determining which partitions are injective for \( G(n) \). We separate out the cases.

**Corollary 2.15.** Suppose \( K \) has characteristic 0. Then a partition \( \lambda \) is injective for \( G(n) \) if and only if \( \lambda^0 \) is an \( l \)-core and \( \text{len}(\bar{\lambda}) \leq \text{stind}_l(\lambda^0) + 1 \).

**Proof.** In this case all \( \hat{G}(n) \)-modules are completely reducible so that \( \bar{\lambda} \) is injective for \( \hat{G}(n) \) and the result follows from Proposition 2.14.

It remains to consider the case in which \( K \) has characteristic \( p > 0 \). A partition \( \lambda \) has unique base \( p \) expansion \( \lambda = \sum_{i \geq 0} p^i \lambda_i \), where each \( \lambda_i \) is a \( p \)-restricted partition. The final results follow immediately from Proposition 2.14.

**Corollary 2.16.** Suppose \( K \) has characteristic \( p > 0 \) and \( q = 1 \). Let \( \lambda \) be a partition with base \( p \) expansion \( \lambda = \sum_{i \geq 0} p^i \lambda_i \). Then \( \lambda \) is injective if and only if each \( \lambda_i \) is a \( p \)-core and \( \text{len}(\bar{\lambda}) \leq \text{stind}_p(\lambda_i) + 1 \), for all \( 0 \leq i < j \).

**Corollary 2.17.** Suppose \( K \) has characteristic \( p > 0 \) and \( q \) is an \( l \)th root of unity, with \( l > 1 \). Let \( \lambda \) be a partition written in standard form \( \lambda = \lambda^0 + l\bar{\lambda} \) and write \( \bar{\lambda} \) in its base \( p \) expansion \( \bar{\lambda} = \sum_{i \geq 0} p^i \bar{\lambda}_i \). Then \( \lambda \) is injective if and only if \( \lambda^0 \) is an \( l \)-core and \( \bar{\lambda}_i \) is a \( p \)-core for each \( i \geq 0 \) and we have \( \text{len}(\bar{\lambda}) \leq \text{stind}_p(\bar{\lambda_i}) + 1 \), for all \( 0 \leq i < j \).

**Examples 2.18.** We finish this section with one example of a partition that is injective and one of a partition that is not for the case in which \( K \) is a field of characteristic 3 and \( q \) is a primitive 4th root of unity. We test these partitions using Corollary 2.17.

(i) Consider first the partition \( \lambda = (20, 9, 6) \). We write \( \lambda \) in the standard form \( \lambda = (8, 5, 2) + 4(3, 1, 1) \). We have that \( (8, 5, 2) \) is a 4-core and the partition \( (3, 1, 1) \) is a 3-core. Moreover \( \text{stind}_4(8, 5, 2) = 2 \) and since \( (3, 1, 1) \) has length 3 we get that \( \lambda = (20, 9, 6) \) is an injective partition.

(ii) Consider now the partition \( \mu = (17, 6, 4) \). We write \( \mu \) in the standard form \( (5, 2) + 4(3, 1, 1) \). We have that \( (5, 2) \) is a 4-core and the partition \( (3, 1, 1) \) is a 3-core. Here, \( \text{stind}_4(5, 2) = 1 \) and since \( (3, 1, 1) \) has length 3 we get that \( \lambda = (17, 6, 4) \) is not an injective partition.
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References


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