TITLE:

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MAXIMAL INEQUALITIES FOR STOCHASTIC CONVOLUTIONS DRIVEN BY COMPENSATED POISSON RANDOM MEASURES IN BANACH SPACES

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Abstract. Let $E$ be a Banach space such that, for some $q \geq 2$, the norm function $x \mapsto \|x\|^q$ is of $C^2$ class and its first and second Fréchet derivatives are bounded by some constant multiples of $(q - 1)$-th power of the norm and $(q - 2)$-th power of the norm. We consider the following stochastic convolution process

$$u(t) = \int_0^t \int_Z S(t-s)\xi(s,z) \tilde{N}(ds,dz), \quad t \geq 0,$$

where $S$ is a $C_0$-semigroup of contractions on $E$ and $\tilde{N}$ is a compensated Poisson random measure. We formulate and prove the following maximal inequality for all $q' \geq q$ and $1 < p \leq 2$,

$$E \sup_{0 \leq s \leq t} |\tilde{u}(s)|_p^{q'} \leq C E \left( \int_0^t \int_Z |\xi(s,z)|_p^p \tilde{N}(ds,dz) \right)^{\frac{q'}{p}},$$

where $\tilde{u}$ is a càdlàg modification of $u$ and $C$ is some positive constant depending on $p$ and $q'$.

Keywords: Stochastic convolution, martingale type $p$ Banach space, Poisson random measure.

AMS Subject Classification: 60H15 (60F10 60H05 60G57 60J75)

1. Introduction

Maximal inequalities for stochastic convolutions in the setting of Hilbert spaces or finite dimensional spaces have received considerable attention for many years. Ichikawa [11] considered maximal inequalities for $C_0$-semigroups of contractions and right continuous martingales in Hilbert spaces, see also Tubaro [30]. A submartingale type inequality for stochastic convolutions of $C_0$-semigroups of contractions and square integrable martingales, also in Hilbert spaces, was obtained by Kotelenez [17]. Kotelenez proved the existence of a càdlàg version of the stochastic convolution process for square integrable càdlàg martingales. In a paper by the second named author and Peszat [5], the authors established a maximal inequality in a certain class of Banach spaces for the stochastic convolution process driven by a Wiener process. Recently, this maximal inequality was generalized by van Neerven and the first named author to $C_0$-contraction semigroups on 2-smooth Banach spaces. Since many results obtained in Wiener case may fail in pure jump type models, maximal inequalities for compensated Poisson random measures deserve an independent investigation. Here we extend the results from [5] to the case where the stochastic convolution is driven.

Date: February 9, 2015.
by a compensated Poisson random measure. We work in the framework of stochastic integrals and
convolutions driven by compensated Poisson random measures recently introduced by the second
and third authors in [3].

Let us now briefly present the content of the paper. In the first section (i.e. section 2) we set up
notations and terminologies and then summarize without proofs some of the standard facts on sto-
chastic integrals with values in martingale type $p$, $p \in (1,2]$, Banach spaces, driven by compensated
Poisson random measures. Section 3 is devoted to the study of the stochastic convolution process
$(u(t))_{t \geq 0}$ driven by a compensated Poisson random measure $\tilde{N}$ which is defined by the following
formula
\begin{equation}
(1.1) \quad u(t) = \int_0^t \int_Z S(t-s)\xi(s,z) \tilde{N}(ds,dz), \quad t \geq 0,
\end{equation}
where $S(t)$, $t \geq 0$ is a $C_0$-semigroup of contractions on a martingale type $p$, $p \in (1,2]$, Banach
space $E$. In particular, we show that there exists a predictable version of the stochastic convolution
process $u$. Under some suitable assumptions we show that the process $u$ is a unique strong solution
to the following stochastic evolution equation
\begin{equation}
(1.2) \quad \frac{d}{dt}u(t) = Au(t) + \int_Z \xi(t,z) \tilde{N}(dt,dz), \quad t \geq 0,
\end{equation}
$u(0) = 0$.

Remark 1.1. It is possible to prove inequality (1.1) by the method based on the Szekőfalvi-Nagy
Theorem on unitary dilations, used earlier in [9], see inequality (4) therein. The latter result has
recently been generalized to Banach spaces of finite cotype by Fröhlich and Weis [7]. However,
this method works only for analytic semigroups of contraction type. The results from the current
paper are valid for all $C_0$-semigroups of contraction type. To be more precise, assume that $E$ is a martingale type $p$ Banach space, with $1 < p \leq 2$, and if we require that $A$ generates an analytic semigroup, then by nearly the same lines as in [9] it would follow that

$$
\mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^p \leq C \mathbb{E} \left( \int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz) \right)^{\frac{p}{p'}}, \quad t \geq 0.
$$

2. Stochastic integral

Let $(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, be a filtered probability space satisfying the usual hypothesis. Let $(S, \mathcal{S})$ be a measurable space. We write $\mathbb{N}$ for the set of all natural numbers and set $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. We denote by $\mathcal{M}_\mathbb{N}(S)$ the space of all $\overline{\mathbb{N}}$-valued measures on $(S, \mathcal{S})$ and $\mathcal{B}(\mathcal{M}_\mathbb{N}(S))$ the smallest $\sigma$-field on $\mathcal{M}_\mathbb{N}(S)$ with respect to which all the mapping $i_B : \mathcal{M}_\mathbb{N}(S) \ni \mu \mapsto \mu(B) \in \mathbb{N}, B \in \mathcal{S}$, are measurable.

**Definition 2.1.** A Poisson random measure on $(S, \mathcal{S})$ over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a map $N : \Omega \to \mathcal{M}_\mathbb{N}(S)$ such that the family $\{N(B) : B \in \mathcal{S}\}$ of $\mathcal{F}$-measurable $\overline{\mathbb{N}}$-valued functions defined by $N(B) := i_B \circ N : \Omega \to \mathbb{N}$ satisfies the following conditions

1. for any $B \in \mathcal{S}$ such that $\eta(B) := \mathbb{E}(N(B)) < \infty$, $N(B)$ is a Poisson random variable with parameter $\eta(B)$, i.e.

   $$
   \mathbb{P}(N(B) = n) = e^{-\eta(B)} \frac{\eta(B)^n}{n!}, \quad n = 0, 1, 2, \cdots;
   $$

2. (independently scattered property) for any pairwise disjoint sets $B_1, \cdots, B_n \in \mathcal{S}$ such that $\eta(B_i) < \infty$, $i = 1, \cdots, n$, the random variables

   $$
   N(B_1), \cdots, N(B_n)
   $$

   are independent.

**Remark 2.2.** In what follows we will often assume that $T \in (0, \infty) \cup \{\infty\}$. Then, if $T = \infty$, by $[0, T]$ we mean the half-line $[0, \infty)$ and $\mathcal{F}_T$ stands for $\mathcal{F}$. Similarly, for $a < \infty$, $(a, \infty]$ (respectively $[a, \infty)$) stands for $(a, \infty)$ (respectively $[a, \infty)$).

**Definition 2.3.** Let us fix $T \in (0, \infty) \cup \{\infty\}$. Let $\mathcal{P}$ denote the $\sigma$-field on $[0, T] \times \Omega$ generated by all left-continuous and $\mathbb{F}$-adapted real valued processes. We call $\mathcal{P}$ the predictable $\sigma$-field. Assume that $(Z, \mathcal{Z})$ is a measurable space. Let $\overline{\mathcal{P}}$ denote the $\sigma$-field on $[0, T] \times \Omega \times Z$ generated by all functions $g : [0, T] \times \Omega \times Z \to \mathbb{R}$ satisfying the following properties

1. for every $t \in [0, T]$, the mapping $(\omega, z) \mapsto g(t, \omega, z)$ is $\mathcal{F}_t \otimes \mathcal{Z}/\mathcal{B}(\mathbb{R})$-measurable,

2. for every $(\omega, z)$, the path $t \mapsto g(t, \omega, z)$ is left-continuous.

We say that an $E$-valued process $g : [0, T] \times \Omega \to E$ is predictable if it is $\mathcal{P}/\mathcal{B}(E)$-measurable.

We say that a function $f : [0, T] \times \Omega \times Z \to E$ is $\mathbb{F} \otimes \mathcal{Z}$-predictable if it is $\overline{\mathcal{P}}/\mathcal{B}(E)$-measurable.
Remark 2.4. The predictable \( \sigma \)-field \( \mathcal{P} \) is also generated by the family \( \mathcal{R} \) (see for instance Th. 3.3 in [18]) defined by

\[
\mathcal{R} = \{ \{0\} \times F : F \in \mathcal{F}_0 \} \cup \{(s, t] \times F : F \in \mathcal{F}_s, 0 \leq s < t, t \in [0, T]\}.
\]

The sets belonging to the family \( \mathcal{R} \) are usually called predictable rectangles. Similarly, one can show, see [34], that the \( \mathcal{F} \otimes \mathcal{Z} \)-predictable \( \sigma \)-field \( \hat{\mathcal{R}} \) is generated by a family \( \hat{\mathcal{R}} \)

\[
\hat{\mathcal{R}} = \{ \{0\} \times F \times B : F \in \mathcal{F}_0, B \in \mathcal{B} \} \cup \{(s, t] \times F \times B : F \in \mathcal{F}_s, B \in \mathcal{Z}, 0 \leq s < t \leq T\}.
\]

Note that a function \( f : [0, T] \times \Omega \times \mathcal{Z} \to E \) which is now called \( \mathcal{F} \otimes \mathcal{Z} \)-predictable, in [34] was called \( \mathcal{F} \)-predictable. We believe that our current terminology is more natural.

Suppose that \((\mathcal{Z}, \mathcal{Z})\) is a measurable space and \(\nu\) is a non-negative \(\sigma\)-finite measure on it. Let \(\text{Leb}\) be the Lebesgue measure on \((\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))\). According to [28], there exists a Poisson random measure \(N\) on \((\mathbb{R}_+ \times \mathcal{Z}, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z})\) with the parameter \(\eta(B) = \mathbb{E}N(B) = \text{Leb} \otimes \nu(B), \) for \(B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z}\). In particular, \(\eta(I \times A) = \text{Leb}(I) \nu(A), \) for \(I \in \mathcal{B}(\mathbb{R}_+)\) and \(A \in \mathcal{Z}\). Here as usual we shall employ the notation

\[
\tilde{N} = N - \text{Leb} \otimes \nu
\]

to denote the compensated Poisson random measure of \(N\).

For \(T \in (0, \infty) \cup \{ \infty \}\), let \(\mathcal{M}^p([0, T] \times \Omega \times \mathcal{Z}; \hat{\mathcal{P}}; E)\) denote the linear space consisting of (equivalence classes of) all \(\mathcal{F} \otimes \mathcal{Z}\)-predictable functions \(f : [0, T] \times \Omega \times \mathcal{Z} \to E\) such that

\[
\int_0^T \int_{\mathcal{Z}} \mathbb{E}|f(t, z)|^p_\nu(\text{d}z) \, \text{d}t < \infty.
\]

In other words, \(\mathcal{M}^p([0, T] \times \Omega \times \mathcal{Z}; \hat{\mathcal{P}}; E)\) is the usual \(L^p\) space of \(E\)-valued functions on \([0, T] \times \Omega \times \mathcal{Z}\) with respect to the \(\sigma\)-field \(\hat{\mathcal{P}}\) and the measure \(\text{Leb} \otimes \mathcal{F} \otimes \nu\). By \(\mathcal{M}^p_{\text{loc}}([0, T] \times \Omega \times \mathcal{Z}; \hat{\mathcal{P}}; E)\) we denote a linear space consisting of all \(\mathcal{F} \otimes \mathcal{Z}\)-predictable functions \(f : [0, \infty) \times \Omega \times \mathcal{Z} \to E\) such that condition (2.1) is satisfied for all \(T > 0\).

Till the end of this section, we will briefly sketch how one constructs, the integral

\[
\int_0^T \int_{\mathcal{Z}} f(t, z) \tilde{N}(\text{d}t, \text{d}z), \text{ for every function } f \in \mathcal{M}^p([0, T] \times \Omega \times \mathcal{Z}; \hat{\mathcal{P}}; E).
\]

This integral we shall call the stochastic integral with respect to the compensated Poisson random measure \(\tilde{N}\). Full details of the definition can be found in [34].

Definition 2.5. A function \(f : [0, T] \times \Omega \times \mathcal{Z} \to E\) is called a step function if there exists a finite sequence of numbers \(0 = t_0 < t_1 < \cdots < t_n = T\) and a finite family \(A^{k}_{j-1}, j = 1, \cdots, n, k = 1, \cdots, m,\) of sets from \(\mathcal{Z}\) with \(\nu(A^{k}_{j-1}) < \infty\) such that

\[
f(t, \omega, z) = \sum_{k=1}^m \sum_{j=1}^n \xi^k_{j-1}(\omega)1_{(t_{j-1}, t_j]}(t)1_{A^k_{j-1}}(z), \quad (t, \omega, z) \in [0, T] \times \Omega \times \mathcal{Z},
\]

where \(\xi^k_{j-1}\) is an \(E\)-valued \(\mathcal{F}_{(t_{j-1}, t_j]}\)-measurable random variable, for every \(j = 1, \cdots, n\) and \(k = 1, \cdots, m,\) and for each \(j = 1, \cdots, n,\) the sets \(A^k_{j-1}, k = 1, \cdots, m,\) are pairwise disjoint.
Note that each step function as defined in Definition 2.5 is $\hat{P}/B(E)$-measurable. In other words, every step function is $F \otimes \mathcal{Z}$-predictable. The class of all step functions satisfying (2.1) will be denoted by $\mathcal{M}^p_{\text{step}}([0, T] \times Z; \hat{P}; E)$.

**Definition 2.6.** Assume that $f$ is a step function in $\mathcal{M}^p_{\text{step}}([0, T] \times Z; \hat{P}; E)$ of the form (2.2). If $t \in [0, T]$, then the stochastic integral over the interval $[0, t]$ of a step function $f$ in $\mathcal{M}^p_{\text{step}}([0, T] \times Z; \hat{P}; E)$ of the form (2.2) with respect to $\tilde{N}$ is a random variable $I_t(f)$, defined by

$$I_t(f) := \sum_{k=1}^m \sum_{j=1}^n \xi^k_{j-1}(\omega)\tilde{N}((t_{j-1} \wedge t, t_j \wedge t) \times A^k_{j-1}).$$

Note that, for every $f \in \mathcal{M}^p_{\text{step}}([0, T] \times Z; \hat{P}; E)$, $I_t(f)$ is linear with respect to $f$ and satisfies the following inequality

$$\mathbb{E}|I_t(f)|^p_E \leq C\mathbb{E} \int_0^t \int_Z |f(s, z)|^p_E \nu(dz) \, ds,$$

where $C$ is the same constant as the one in the martingale-type-$p$ property of the space $E$. Moreover, the process $I_t(f)$, $t \geq 0$ is an $E$-valued, mean 0 and càdlàg $\hat{F}$-martingale.

Let us describe now how this definition can be extended to all functions in $\mathcal{M}^p([0, T] \times Z; \hat{P}; E)$. Take $f \in \mathcal{M}^p([0, T] \times Z; \hat{P}; E)$. Then we can find a sequence $\{f^n\}_{n=1}^{\infty}$ of functions in $\mathcal{M}^p_{\text{step}}([0, T] \times Z; \hat{P}; E)$, see Theorem 3.2.23 in [34], such that

$$\mathbb{E} \int_0^T \int_Z |f(t, \omega, z) - f^n(t, \omega, z)|^p_E \nu(dz) \, dt \to 0, \quad \text{as } n \to \infty.$$

It follows from (2.3) that

$$\mathbb{E}|I_T(f^n) - I_T(f^m)|^p_E \leq C\mathbb{E} \int_0^T \int_Z |f^n(s, z) - f^m(s, z)|^p_E \nu(dz) \, ds \to 0,$$

as $n, m \to \infty$. In other words, $\{I_T(f^n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^p(\Omega, E, \mathcal{F}_T)$. Thus this sequence $\{I_T(f^n)\}_{n=1}^{\infty}$ of random variables will converge in $L^p(\Omega, \mathcal{F}_T; E)$ to some particular random variable which we shall denote by $I_T(f)$ or $\int_0^T \int_Z f(s, z)\tilde{N}(ds, dz)$. Moreover, it does not depend on the choice of the sequence $\{f^n\}_{n=1}^{\infty}$ of approximating step functions. We usually call $I_T(f)$ the stochastic integral of $f$ with respect to the compensated Poisson random measure $\tilde{N}$. For $0 \leq a \leq b \leq T$, $B \in Z$ and $f \in \mathcal{M}^p([0, T] \times Z; \hat{P}; E)$, since $1_{(a,b)}1_B f$ is also in $\mathcal{M}^p([0, T] \times Z; \hat{P}; E)$, we can define the stochastic integral from $a$ to $b$ of the function $f \in \mathcal{M}^p([0, T] \times Z; \hat{P}; E)$ by

$$I_{a,b}^B(f) = \int_a^b \int_B f(s, z)\tilde{N}(ds, dz) = I_T(1_{(a,b)}1_B f).$$

For simplicity, we denote $I_t(f) := I_T(1_{[0,t]} f)$, for $t \in (0, T]$ and set $I_t(f) = 0$, when $t = 0$. Let $f \in \mathcal{M}^p([0, T] \times Z; \hat{P}; E)$. It was shown in [3] and [34] (see also [26] for the case $p = 2$) that the process $I_t(f)$, $t \in [0, T]$ is a càdlàg $p$-integrable $\hat{F}$-martingale with mean 0. In particular, $I_t(f)$ has a modification which has $\mathbb{P}$-a.s. càdlàg trajectories and satisfies the following inequality

$$\mathbb{E}|I_t(f)|^p_E = \mathbb{E} \int_0^t \int_Z |f(s, z)|^p_E \nu(dz) \, ds \leq C\mathbb{E} \int_0^t \int_Z |f(s, z)|^p_E \nu(dz) \, ds.$$
From now on, while considering the stochastic process \( \int_0^T \int_Z f(s, z) \tilde{N}(ds, dz) \), \( t \in [0, T] \), we will assume that it has \( \mathbb{P} \)-a.s. càdlàg trajectories.

**Remark 2.7.** We have defined the stochastic integral for integrands belonging to the class \( \mathcal{M}^p([0, T] \times Z; \tilde{\mathcal{P}}; E) \) of predictable processes. One should note that in the paper [3] by the second and third authors, the integral is defined for an analogous class of progressively measurable processes. See also [26]. This approach is also discussed in section 5 of the present paper.

If \( \tau \) is a stopping time with \( \mathbb{P}\{\tau \leq T\} = 1 \), we may set, for \( \omega \in \Omega \),

\[ I_\tau(f)(\omega) = I_t(f)(\omega) \quad \text{with} \quad t = \tau(\omega). \]

(2.6)

Analogously, we shall also use the notation \( I_x(f) =: \int_0^T \int_Z f(s, z) \tilde{N}(ds, dz) \). In this case, one can show that

\[ \int_0^T \int_Z f(s, z) \tilde{N}(ds, dz) = \int_0^T \int_Z 1_{[0, \tau]}(s)f(s, z) \tilde{N}(ds, dz), \quad \mathbb{P}\text{-a.s.} \]

(2.7)

**Remark 2.8.** Note that since \( \tau \) is a stopping time (without any additional property), the random process

\[ 1_{(0, \tau]} : [0, \infty) \times \omega \ni (s, \omega) \mapsto 1_{0, \tau}(\omega)(s) \in [0, 1] \]

is predictable, see the comment after [24, Definition IV.5.3] or Proposition 4.6 in [18]. Hence, provided that the process \( f \) belonging to \( \mathcal{M}^p([0, T] \times Z; \tilde{\mathcal{P}}; E) \), the process \( 1_{[0, \tau]}f \) belongs to that space as well. In particular, the integral on the RHS of (2.7) is well defined.

Indeed, (2.7) can be easily verified for step functions \( f^n \) of the form (2.2) that

\[ \int_0^T \int_Z 1_{(0, \tau]}(s)f^n(s, z) \tilde{N}(ds, dz) = \int_0^T \int_Z f^n(s, z) \tilde{N}(ds, dz), \quad \mathbb{P}\text{-a.s.} \]

(2.8)

Take \( f \in \mathcal{M}^p([0, T] \times Z; \tilde{\mathcal{P}}; E) \). Then as we discussed before, there exists an \( \mathcal{M}_{\text{step}}^p((0, T] \times Z; \tilde{\mathcal{P}}; E) \)-valued sequence \( \{f^n\} \) such that

\[ \mathbb{E} \int_0^T \int_Z 1_{(0, \tau]}(s)|f(t, z) - f^n(s, z)|_E^p \nu(dz)dt \to 0, \quad \text{as} \ n \to \infty. \]

By applying inequality (2.5), we infer

\[ \lim_{n \to \infty} \mathbb{E} \left| \int_0^T \int_Z 1_{(0, \tau]}(s)f^n(s, z) \tilde{N}(ds, dz) - \int_0^T \int_Z 1_{(0, \tau]}(s)f(s, z) \tilde{N}(ds, dz) \right|_E^p = 0. \]

Hence, we can extract a subsequence (denoted again by the same notation for simplicity) such that

\[ \lim_{n \to \infty} \int_0^T \int_Z 1_{(0, \tau]}(s)f^n(s, z) \tilde{N}(ds, dz) = \int_0^T \int_Z 1_{(0, \tau]}(s)f(s, z) \tilde{N}(ds, dz), \quad \mathbb{P}\text{-a.s.} \]

(2.9)

On the other hand, since \( I_t(f) - I_t(f^n) \) is a right-continuous martingale, \( |I_t(f) - I_t(f^n)|_E \) is a real-valued submartingale. Since \( \mathbb{P}(\tau \leq T) = 1 \), by the stopped Doob inequality, we have

\[ \mathbb{E}|I_\tau(f) - I_\tau(f^n)|_E^p \leq \left( \frac{p}{p-1} \right)^p \mathbb{E}|I_T(f) - I_T(f^n)|_E^p \to 0, \quad \text{as} \ n \to \infty. \]

\[ \text{as} \ n \to \infty. \]
It follows (by selecting a further subsequence) that
\begin{equation}
\lim_{n \to \infty} \int_0^T \int_Z f^n(s, z) \tilde{N}(ds, dz) = \int_0^T \int_Z f(s, z) \tilde{N}(ds, dz), \ \mathbb{P}\text{-a.s.}
\tag{2.10}
\end{equation}
Hence, (2.7) follows directly from (2.8), (2.9) and (2.10).

In particular, for every \( t \in [0, T] \) and any stopping time \( \tau \) with \( \mathbb{P}(\tau < \infty) = 1 \), we have
\begin{equation}
\int_0^{t \wedge \tau} \int_Z f(s, z) \tilde{N}(ds, dz) = \int_0^t \int_Z 1_{(0, \tau]}(s) f(s, z) \tilde{N}(ds, dz).
\tag{2.11}
\end{equation}
We will also need the following result about the integral \( \int_0^t \int_Z f(s, \omega, z) N(ds, dz)(\omega) \), which is defined, for every \( \omega \in \Omega \), as Bochner integral with respect to measure \( N(ds, dz)(\omega) \) on \([0, t] \times Z\).

Later on we will use in the proof the notion of point processes (see [12]). For more details we refer the reader to [4] or [34]. Let us assume that \( \pi \) is a stationary Poisson point process on \((Z, \mathcal{Z})\) with the intensity measure \( \nu \). One can check Theorem 5.4 in [25] for the existence of such Poisson point process \( \pi \). For simplicity of notation, the Poisson random measure associated to the Poisson point process \( \pi \) will be still denoted by \( N \). We use the notation \( \tilde{N}(t, A) = N(t, A) - t\nu(A), t \geq 0, A \in \mathcal{Z} \) to denote its compensated Poisson random measure.

**Proposition 2.9.** If \( T \in (0, \infty) \cup \{ \infty \} \) and \( f : [0, T] \times \Omega \times Z \to E \) is a \( \mathcal{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathcal{Z} \)-measurable function and
\begin{equation}
\mathbb{E} \int_0^T \int_Z |f(s, z)|_E N(ds, dz) < \infty,
\tag{2.12}
\end{equation}
then we have for every \( t \in [0, T] \),
\begin{equation}
\int_0^t \int_Z f(s, \omega, z) N(ds, dz)(\omega) = \sum_{s \leq t} f(s, \omega, \pi(s, \omega)), \ \mathbb{P} \text{-a.s.}
\tag{2.13}
\end{equation}

**Proof.** Since \( f \) is \( \mathcal{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathcal{Z} \)-measurable, for every \( \omega \in \Omega \), \( f(\cdot, \omega, \cdot) \) is \( \mathcal{B}([0, T]) \otimes \mathcal{Z} \)-measurable. By (2.12), we deduce that \( \int_0^T \int_Z |f(s, \omega, z)|_E N(ds, dz) < \infty, \mathbb{P}\text{-a.s.} \). Let us choose for the remainder of the proof an \( \omega \in \Omega \) such that the previous integral is finite. Hence, \( f(\cdot, \omega, \cdot) \) is Bochner integrable with respect to \( N(ds, dz)(\omega) \). Moreover, we can find a sequence \( \{f^n\} \) of functions on \([0, T] \times Z\) of the form \( \sum_{i=1}^m x_i B_i \), \( x_i \in E \) and \( B_i \in \mathcal{B}([0, T]) \otimes \mathcal{Z} \) such that \( |f^n(t, \omega, z) - f(t, \omega, z)|_E \) decreases to 0 as \( n \to \infty \), for all \((t, z) \in [0, T] \times Z\). Hence it is enough to show (2.13) for functions of the form \( \sum_{i=1}^m a_i B_i \). For this, observe that
\begin{align*}
\int_0^t \int_Z f(s, \omega, z) N(ds, dz)(\omega) &= \sum_{i=1}^m x_i N(B_i)(\omega) \\
&= \sum_{i=1}^m x_i \sum_{s \in [0,t] \cap \partial(\pi(\omega))} 1_{B_i}(s, \pi(s, \omega)) \\
&= \sum_{s \in [0,t] \cap \partial(\pi(\omega))} \sum_{i=1}^m x_i 1_{B_i}(s, \pi(s, \omega)) \\
&= \sum_{s \in [0,t] \cap \partial(\pi(\omega))} f(s, \omega, \pi(s, \omega)).
\end{align*}
3. Stochastic convolution

In this section, we continue to assume that \( E \) is a separable Banach space of martingale type \( p \), where \( p \in (1, 2] \). Let \( (S(t))_{t \geq 0} \) be a contraction \( C_0 \)-semigroup on \( E \) with the infinitesimal generator \( A \). Let us denote by \( R(\lambda, A) = (\lambda I - A)^{-1}, \lambda > 0 \), the resolvent operator of \( A \) and by \( A_\lambda = A(\lambda I - A)^{-1} \) the Yosida approximation of \( A \). It is well known that \( A_\lambda \) is a bounded operator on \( E \) and \( A_\lambda x \to x \), as \( \lambda \to \infty \), for \( x \in A \) and \( \lambda R(\lambda, A)x \to x \), as \( \lambda \to \infty \), for all \( x \in E \). Moreover, \( \lambda R(\lambda, A)x \in D(A) \), for all \( x \in E \).

Suppose that \( \xi \in M_{\text{loc}}^p([0, T] \times Z; \hat{\mathbb{P}}; D(A)) \) and \( \tilde{N} \) is a compensated Poisson random measure corresponding to the point process \( \pi = (\pi(t))_{t \geq 0} \). The aim of this paper is to study the path properties of the stochastic convolution process \( u \) defined by

\[
(3.1) \quad u(t) = \int_0^t \int_Z S(t-s)\xi(s, z) \tilde{N}(ds, dz), \quad t \geq 0.
\]

Now let us consider Problem (1.2), which for the convenience of the reader we rewrite below.

\[
(3.2) \quad \frac{d}{dt} u(t) = Au(t) dt + \int_Z \xi(t, z) \tilde{N}(dt, dz), \quad t \geq 0, \quad u(0) = 0.
\]

**Definition 3.1.** Suppose that \( \xi \in M_{\text{loc}}^p([0, T] \times Z; \hat{\mathbb{P}}; D(A)) \). A strong solution to Problem (3.2) on the time interval \([0, T]\) is a \( D(A) \)-valued \( \mathbb{F} \)-adapted stochastic process \((u(t))_{t \in [0, T]}\) with \( E \)-valued c\( \dot{\text{a}} \)dl\( \dot{\text{a}} \)g trajectories such that

1. \( u(0) = 0 \) a.s.
2. For any \( t \in [0, T] \) the following equality holds \( \mathbb{P} \)-a.s.

\[
(3.3) \quad u(t) = \int_0^t Au(s) ds + \int_0^t \int_Z \xi(s, z) \tilde{N}(ds, dz).
\]

Similarly we can define a strong solution to Problem (3.2) if \( T = \infty \) and \( \xi \in M_{\text{loc}}^p([0, T] \times Z; \hat{\mathbb{P}}; D(A)) \).

**Lemma 3.2.** Let \( \xi \in M_{\text{loc}}^p([0, T] \times Z; \hat{\mathbb{P}}; D(A)) \). Then the process \( u \) defined by

\[
(3.4) \quad u(t) = \int_0^t \int_Z S(t-s)\xi(s, z) \tilde{N}(ds, dz), \quad t \geq 0,
\]

is a unique strong solution of Equation (1.2). In particular, \( u \) has \( E \)-valued c\( \dot{\text{a}} \)dl\( \dot{\text{a}} \)g trajectories.

**Proof.** Let us fix \( T > 0 \) and \( t \in [0, T] \). Define a function \( F : [0, t] \times E \ni (s, x) \mapsto S(t-s)x \in E \). It is straightforward to see that the function \( F \) is separately continuous. Moreover, since \([0, t]\) is compact, we infer that \( F \) is continuous. This, together with the \( \mathbb{F} \otimes Z \)-predictability assumption on \( \xi \), implies that the composition mapping

\[
[0, t] \times \Omega \times Z \ni (s, \omega, z) \mapsto (s, \xi(s, \omega, z)) \mapsto F(s, \xi(s, \omega, z)) \in E
\]


}\]
is $F \otimes Z$-predictable. On the other hand, since each operator $S(t)$, $t \geq 0$ is a contraction on $E$ and $\xi$ is in $M^p([0,T] \times Z; \hat{P}; E)$, we have

$$\mathbb{E} \int_0^T |1_{(0,t)}(s)S(t-s)\xi(s,z)|_p^p \nu(dz) \, ds \leq \mathbb{E} \int_0^T |\xi(s,z)|_p^p \nu(dz) \, ds < \infty.$$ 

Therefore, the process $S(t-s)\xi(s,z)$ belongs to $M^p([0,T] \times Z; \hat{P}; E)$. Hence, since the number $t$ is fixed, the process defined by

$$\int_0^T \int Z S(t-s)\xi(s,z) \tilde{N}(ds,dz), \quad r \in [0,t],$$

is a $F$-martingale on $[0,t]$, see [34]. In particular, for each $r \in [0,t]$, the random variable $\int_0^r \int Z S(t-s)\xi(s,z)\tilde{N}(ds,dz)$ is $F_r$-measurable and hence $u(t)$ is $F_r$-measurable.

Now we proceed to show that $u(t) \in D(A)$. Since $\xi \in M^p([0,T] \times Z; \hat{P}; D(A))$ and $R(\lambda, A)A = \lambda R(\lambda, A) - I_E$ on $D(A)$, we obtain

$$R(\lambda, A) \int_0^t \int Z A S(t-s)\xi(s,z) \tilde{N}(ds,dz) = \lambda R(\lambda, A) \int_0^t \int Z S(t-s)\xi(s,z) \tilde{N}(ds,dz)$$

$$\quad - \int_0^t \int Z S(t-s)\xi(s,z) \tilde{N}(ds,dz).$$

Hence, it follows that

$$\int_0^t \int Z S(t-s)\xi(s,z) \tilde{N}(ds,dz)$$

$$= R(\lambda, A) \left[ \lambda \int_0^t \int Z S(t-s)\xi(s,z) \tilde{N}(ds,dz) - \int_0^t \int Z A S(t-s)\xi(s,z) \tilde{N}(ds,dz) \right].$$

Since the range of $R(\lambda, A)$ is equal to $D(A)$, we infer that $u(t) \in D(A)$.

Next we shall show that

$$(3.5) \quad A \int_0^t \int Z S(t-s)\xi(s,z) \tilde{N}(ds,dz) = \int_0^t \int Z A S(t-s)\xi(s,z) \tilde{N}(ds,dz), \quad \mathbb{P}\text{-a.s.}$$

For this, let us take $h \in (0,t)$ and observe that since the operator $\frac{S(h)-I}{h}$ is bounded, we get the following equality

$$\frac{S(h)-I}{h} \int_0^t \int Z S(t-s)\xi(s,z) \tilde{N}(ds,dz) = \int_0^t \int Z \frac{S(h)-I}{h} S(t-s)\xi(s,z) \tilde{N}(ds,dz).$$
So by applying the triangle inequality and (2.5), we find
\[
\mathbb{E} \left| A \int_0^t \int_Z S(t-s)\xi(s,z) \tilde{N}(ds,dz) - \int_0^t \int_Z AS(t-s)\xi(s,z) \tilde{N}(ds,dz) \right|_E^p \\
\leq 2^p \mathbb{E} \left| A \int_0^t \int_Z S(t-s)\xi(s,z) \tilde{N}(ds,dz) - \frac{S(h) - I}{h} \int_0^t \int_Z S(t-s)\xi(s,z) \tilde{N}(ds,dz) \right|_E^p \\
+ 2^p \mathbb{E} \left| \int_0^t \int_Z AS(t-s)\xi(s,z) \tilde{N}(ds,dz) - \int_0^t \int_Z \frac{S(h) - I}{h} S(t-s)\xi(s,z) \tilde{N}(ds,dz) \right|_E^p \\
\leq 2^p \mathbb{E} \left| A - \frac{S(h) - I}{h} \right| u(t) \right|_E^p \\
+ C_p \mathbb{E} \int_0^t \int_Z \left| AS(t-s)\xi(s,z) - \frac{1}{h} (S(h) - I) S(t-s)\xi(s,z) \right|_E^p \nu(dz) ds
\]
(3.6) =: I(h) + II(h).

First we will deal with II(h). Since \(\xi(s,z) \in \mathcal{D}(A)\), we observe that
\[
\frac{S(h) - I}{h} S(t-s)\xi(s,z) = \frac{1}{h} \int_0^h S(r)AS(t-s)\xi(s,z) \, dr.
\]
So by using the uniform condition of the operators \(S(h)\) and \(S(t-s)\), we deduce that
\[
\left| \frac{S(h) - I}{h} S(t-s)\xi(s,z) \right|_E \leq C|A\xi(s,z)|_E.
\]
Hence we infer that the integrand of II(h) is bounded by a function \(C_1|A\xi(s,z)|_E\) which belongs to \(\mathcal{M}^p([0,T] \times Z; \mathbb{R})\) by assumption. Clearly, the integrand
\[
\left| AS(t-s)\xi(s,z) - \frac{1}{h} (S(h) - I) S(t-s)\xi(s,z) \right|_E^p
\]
converges to 0 pointwise on \([0,t] \times \Omega \times Z\). Therefore, by the Lebesgue Dominated Convergence Theorem (LDCT for short), II(h) converges to 0 as \(h \searrow 0\).

Now we turn our attention to the term I(h). Since \(\mathbb{E}\|u(t)\|_{\mathcal{D}(A)}^p < \infty\) and \(\|\frac{1}{h}(S(h) - I)x\|_E \leq |Ax|_E\), for all \(x \in \mathcal{D}(A)\), we infer by applying the LDCT that I(h) converges to 0 as \(h \searrow 0\) as well. Hence (3.5) holds.

In order to finish the proof, we need to verify (3.3). By (3.5) and the Fubini Theorem we find
\[
\int_0^t Au(s) \, ds = \int_0^t \int_0^s \int_Z AS(s-r)\xi(r,z) \tilde{N}(dr,dz) \, ds \\
= \int_0^t \int_Z \int_0^t AS(s-r)\xi(r,z) \, ds \tilde{N}(dr,dz) \\
= \int_0^t \int_Z (S(t-r)\xi(r,z) - \xi(r,z)) \tilde{N}(dr,dz) \\
= u(t) - \int_0^t \int_Z \xi(r,z) \tilde{N}(dr,dz), \quad \mathbb{P}\text{-a.s.}
\]
Hence we have
\[
u(t) = \int_0^t Au(s) \, ds + \int_0^t \int_Z \xi(s,z) \tilde{N}(ds,dz)
\]
from which we can also see that \( u \) is a càdlàg process.

For the uniqueness, suppose that \( u^1 \) and \( u^2 \) are two strong solutions of Problem (1.2). Let \( w = u^1 - u^2 \). Then we infer
\[
\begin{align*}
    w(t) &= u^1(t) - u^2(t) = \int_0^t A(u^1(s) - u^2(s)) \, ds = A \int_0^t w(s) \, ds.
\end{align*}
\]

Put \( v(t) = \int_0^t w(s) \, ds \). Then \( v(t) \) is continuously differentiable on \([0, T]\) and \( v(t) \in D(A)\). Now applying the Itô formula to the function \( f(s) = S(t-s)v(s) \) yields
\[
\begin{align*}
    \frac{df(s)}{ds} &= -AS(t-s)v(s) + S(t-s)\frac{dv(s)}{ds} \\
    &= -AS(t-s)v(s) + S(t-s)w(s) = -AS(t-s)v(s) + S(t-s)Av(s) = 0.
\end{align*}
\]

So we infer \( v(t) = f(t) = f(0) = S(t)v(0) = 0 \) a.s.. Therefore, \( w(t) = 0 \) a.s.. That is \( u^1(t) = u^2(t) \) a.s. \( t \geq 0 \).

\[\square\]

4. Maximal inequalities for stochastic convolution

From now on we make the following assumption on the Banach space \( E \).

**Assumption 4.1.** Suppose that \( E \) is a real separable Banach space. In addition we assume that the Banach space \( E \) satisfies the following condition:
There exists an equivalent norm \( \cdot \| \cdot \|_E \) on \( E \) and \( q \in [p, \infty) \) such that the function \( \phi : \mathbb{R} \ni x \mapsto \|x\|^q_E \in \mathbb{R} \), is of class \( C^2 \) and there exist constants \( k_1, k_2 \) such that for every \( x \in E \), \( \|\phi'(x)\| \leq k_1 \|x\|^{q-1}_E \) and \( \|\phi''(x)\| \leq k_2 \|x\|^{q-2}_E \).

Now we proceed with the study of the stochastic convolution
\[
(4.1) \quad u(t) = \int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz), \ t \geq 0.
\]

**Remark 4.2.** It can be proved, see Appendix, that if the real separable Banach space \( E \) satisfies Assumption (4.1), then \( E \) is of martingale type \( p \), for all \( p \in (1, 2] \). So the stochastic convolution (4.1) is well defined for \( \xi \in \mathcal{M}^p([0, T] \times Z; \mathcal{F} ; E) \) and in particular, if \( \xi \in \mathcal{M}^p([0, T] \times Z; \mathcal{F} ; D(A)) \), then (4.1) is a unique strong solution of equation (1.2) by Lemma 3.2.

**Remark 4.3.** Notice that Sobolev spaces \( H^{s,r} \) with \( r \in [2, \infty) \) and \( s \in \mathbb{R} \) satisfy Assumption (4.1) and \( L^r \)-spaces with \( r \geq 2 \) also satisfies Assumption (4.1).

Before proving the main theorem, we first need the following Lemmas.

**Lemma 4.4.** For all \( x \in D(A) \), \( \phi'(x)(Ax) \leq 0 \).

**Proof.** This follows immediately from the fact that the function \( t \mapsto \phi(S(t)x) \) is decreasing and
\[
\begin{align*}
    \frac{d\phi(S(t)x)}{dt} \bigg|_{t=0} &= \phi'(S(0)x)(Ax) = \phi'(x)(Ax).
\end{align*}
\]
\[\square\]
Lemma 4.5. There exists a version $\tilde{u}$ of the process $u$ such that the function $\sup_{t \geq 0} |\tilde{u}(t)|$ is $\mathcal{F}_T$-measurable.

Proof. According to Remark 4.2, we can take $p \in (1, 2]$. We begin with showing that the process $u$ is continuous in the $p$-mean. By applying the inequality $|a + b|^p \leq 2^p |a|^p + 2^p |b|^p$, inequality (2.5) and the contraction property of the semigroup $S(t)$, $t \geq 0$, we have, for $0 \leq r < t \leq T$,

$$
\mathbb{E}|u(t) - u(r)|^p \leq \mathbb{E} \left| \int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) - \int_0^r \int_Z S(r-s)\xi(s,z)\tilde{N}(ds,dz) \right|^p
$$

\begin{align*}
&\leq 2^p \mathbb{E} \left| \int_r^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) \right|^p \\
&\quad + 2^p \mathbb{E} \left| \int_0^r \int_Z \left( S(t-s) - S(r-s) \right)\xi(s,z)\tilde{N}(ds,dz) \right|^p \\
&\leq 2^p C_p \mathbb{E} \int_r^t \int_Z |S(t-s)\xi(s,z)|^p \nu(dz) \, ds \\
&\quad + 2^p C_p \mathbb{E} \int_0^r \int_Z \left| (S(t-s) - S(r-s))\xi(s,z) \right|^p \nu(dz) \, ds \\
&\leq 2^p C_p \mathbb{E} \int_0^T \int_Z 1_{[r,t]}(s)\xi(s,z)|^p \nu(dz) \, ds \\
&\quad + 2^p C_p \mathbb{E} \int_0^T \int_0^r \left| 1_{[0,r]}(S(t-s) - S(r-s))\xi(s,z) \right|^p \nu(dz) \, ds.
\end{align*}

Here we observe that $1_{[r,t]}(s)\xi(s,z)|^p \nu(dz)$ converges to 0 for all $(s, \omega, z) \in [0, T] \times \Omega \times Z$, as $t \downarrow r$ or $r \uparrow t$. So by the LDCT, the first term on the right hand side of the above inequality converges to 0 as $t \downarrow r$ or $r \uparrow t$. For the second term, by the continuity of $C_0$-semigroup $S(t)$, $t \geq 0$, the integrand $1_{(0,r]}(S(t-s) - S(r-s))\xi(s,z)$ converges to 0 pointwise on $[0, T] \times \Omega \times Z$, as $t \downarrow r$ or $r \uparrow t$. Moreover we find

$$
\sup_{t \geq 0} |\tilde{u}(t)| = \sup_{t \in [0,T]} \lim_{s_n \rightarrow t, s_n \in T_0} |\tilde{u}(s_n)| = \sup_{s_n \in T_0} |\tilde{u}(s_n)|,
$$

where $\sup_{s_n \in T_0} |\tilde{u}(s_n)|$ is $\mathcal{F}_T$-measurable. Therefore, the function $\sup_{t \geq 0} |\tilde{u}(t)|$ is also $\mathcal{F}_T$-measurable.

\[\square\]

It is worth pointing out that since by Proposition 3.6 in [6], every adapted and stochastically continuous process on an interval $[0, T]$ has a predictable version on $[0, T]$, we conclude that the process $u(t)$, $t \geq 0$ has a predictable version. Henceforth, when we study the stochastic convolution process $u$, we refer to the version of $u$ that is càdlàg and its supremum over every compact interval $[0, T]$ is $\mathcal{F}_T$-measurable.
Theorem 4.6. Suppose that $E$ is a real separable Banach space satisfying Assumption 4.1. Assume that $\xi \in M^p_{\text{loc}}([0,T] \times Z; \hat{P}; E)$. Then there exists a separable and càdlàg modification $\tilde{u}$ of $u$ and a constant $C$ such that for every stopping time $\tau > 0$ and every $T > 0$,

$$
\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau} |\tilde{u}(t)|^q_E \leq C \mathbb{E} \left( \int_0^{T \wedge \tau} \int_Z |\xi(s,z)|^p_E \tilde{N}(ds,dz) \right)^{\frac{q'}{p}} ,
$$

where $1 < p \leq 2$, $q' \geq q$ and $q$ is the number from Assumption 4.1.

Proof. Let us fix $T > 0$ and in view of Remark 4.2, let us also fix $p \in (1,2]$. 

Case I. We first prove (4.2) for $\xi \in M^p([0,T] \times Z; \hat{P}; D(A))$. We have shown in Lemma 3.2 that the process $u$ is a unique strong solution to Problem (3.2) satisfying

$$
u(t) = \int_0^t A \nu(s) \, ds + \int_0^t \int_Z \xi(s,z) \tilde{N}(ds,dz), \ t \in [0,T].
$$

Since the function $\phi : E \ni x \mapsto |x|^q_E$ is of $C^2$ class by assumption, one may apply the Itô formula (see [10]) to the process $u$ given by (4.3) and get, for $t \geq 0$,

$$
\phi(u(t)) = \int_0^t \phi'(u(s))(Au(s)) \, ds + \int_0^t \int_Z \phi'(u(s-))(\xi(s,z)) \tilde{N}(ds,dz)
+ \int_0^t \int_Z \left[ \phi(u(s-)+\xi(s,z)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s,z)) \right] \tilde{N}(ds,dz) \ \mathbb{P}\text{-a.s.}
$$

Let $\tau \geq 0$ be a stopping time. Since by Lemma 4.4, $\phi'(x)(Ax) \leq 0$, for all $x \in D(A)$, we infer that for $t \geq 0$,

$$
\phi(u(t \wedge \tau)) \leq \int_0^t \int_Z 1_{(0,\tau]}(s) \phi'(u(s-))(\xi(s,z)) \tilde{N}(ds,dz)
+ \int_0^{t \wedge \tau} \int_Z \left[ \phi(u(s-)+\xi(s,z)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s,z)) \right] \tilde{N}(ds,dz)
=: I_1(t) + I_2(t) \ \mathbb{P}\text{-a.s.}
$$
Note that $I_1(t)$ is an $\mathbb{R}$-valued local martingale. Applying the real-valued version of Burkholder-Davis-Gundy inequality (see [15]) to $I_1$ we deduce for some constant $C$ that

\[
\mathbb{E} \sup_{0 \leq t \leq T} |I_1(t)|_E \leq C \mathbb{E} \left( \int_0^T \int_Z 1_{(0, \tau]}(s) |\phi'(u(s-))(\xi(s, z))|^2_E N(ds, dz) \right)^{\frac{1}{2}} \\
= C \mathbb{E} \left( \sum_{t \leq T} 1_{(0, \tau]}(s) |\phi'(u(s-))(\xi(s, \pi(s)))|^2_E \right)^{\frac{1}{2}} \\
\leq C \mathbb{E} \left( \sum_{t \leq T} 1_{(0, \tau]}(s) |\phi'(u(s-))(\xi(s, \pi(s)))|^p_E \right)^{\frac{1}{p}} \\
= C \mathbb{E} \left( \int_0^T \int_Z 1_{(0, \tau]}(s) |\phi'(u(s-))(\xi(s, z))|^p_E N(ds, dz) \right)^{\frac{1}{p}} \\
\leq k_1 C \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau} |u(t)|^{q-1}_E \left( \int_0^{T \wedge \tau} \int_Z |\xi(s, z)|^p_E N(ds, dz) \right)^{\frac{1}{p}}.
\]

To estimate the integral $I_2(t)$, we observe first that for every $t \geq 0$,

\[
|I_2(t)|_E \leq \int_0^{t \wedge \tau} \int_Z \left| \phi(u(s-) + \xi(s, z)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s, z)) \right| N(ds, dz) = \sum_{s \in (0, t \wedge \tau] \cap D(\pi)} \left| \phi(u(s-) + \xi(s, \pi(s))) - \phi(u(s-)) - \phi'(u(s-))(\xi(s, \pi(s))) \right|_E, \quad \mathbb{P}\text{-a.s.}
\]

Let us recall that by the assumption, the function $\phi$ is of $C^2$ class. Applying the mean value Theorem, see [16], to the function $\phi$, for each $s \in [0, t \wedge \tau]$ we can find $0 < \theta < 1$ such that

\[
\left| \phi(u(s-) + \xi(s, \pi(s))) - \phi(u(s-)) - \phi'(u(s-))(\xi(s, \pi(s))) \right|_E = |\xi(s, \pi(s))|_E \left| \phi'(u(s-) + \theta \xi(s, \pi(s))) \right|_{\mathcal{L}(E)}.
\]

By the assumptions $|\phi'(x)| \leq k_1 |x|^{q-1}_E$, $x \in E$ and the fact that $|x + y|_E \leq \max\{|x|_E, |x + y|_E\}$ for all $x, y \in E$, we obtain

\[
\left| \phi'(u(s-) + \theta \xi(s, \pi(s))) \right|_{\mathcal{L}(E)} \leq k_1 \left| u(s-) + \theta \xi(s, \pi(s)) \right|^{q-1}_E \\
\leq k_1 \max \left\{ \left| u(s-) \right|^{q-1}_E, \left| u(s-) + \xi(s, \pi(s)) \right|^{q-1}_E \right\}.
\]

Observe that for all $0 \leq s \leq t \wedge \tau$,

\[
|u(s-)|^q_E \leq \sup_{0 \leq r \leq t \wedge \tau} |u(r-)|^{q-1}_E \leq \sup_{0 \leq t \leq T \wedge \tau} |u(t)|^{q-1}_E.
\]

Moreover, since $u(s-) + \xi(s, \pi(s)) = u(s)$, we get

\[
|u(s-) + \xi(s, \pi(s))|^q_E \leq \sup_{0 \leq r \leq t \wedge \tau} |u(r)|^{q-1}_E \leq \sup_{0 \leq t \leq T \wedge \tau} |u(t)|^{q-1}_E.
\]

Therefore, we infer that for each $s \in [0, t \wedge \tau]$,

\[
\left| \phi(u(s-) + \xi(s, \pi(s))) - \phi(u(s-)) \right|_E \leq k_1 |\xi(s, \pi(s))|_E \sup_{0 \leq t \leq T \wedge \tau} |u(t)|^{q-1}_E.
\]
It follows that
\[
\left| \phi(u(s-)) + \xi(s, \pi(s)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s, \pi(s))) \right|_E \\
\leq \left| \phi(u(s-)) + \xi(s, \pi(s)) - \phi(u(s-)) \right|_E + \left| \phi'(u(s-))(\xi(s, \pi(s))) \right|_E \\
\leq 2k_1|\xi(s, \pi(s))|_E \sup_{0 \leq t \leq T \wedge \tau} |u(t)|^{q-1}.
\]

On the other hand, we can also find some \(0 < \delta < 1\) such that
\[
\left| \phi(u(s-)) + \xi(s, \pi(s)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s, \pi(s))) \right|_E \\
\leq \frac{1}{2} |\xi(s, \pi(s))|_E^2 |\phi''(u(s-)) + \delta \xi(s, \pi(s)))| \leq \frac{k_2}{2} |\xi(s, \pi(s))|_E^2 |u(s-)) + \delta \xi(s, \pi(s))|^{q-2}.
\]
Hence, a similar argument as above, we infer that for \(s \in [0, t \wedge \tau]\)
\[
\left| \phi(u(s-)) + \xi(s, \pi(s)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s, \pi(s))) \right|_E \leq \frac{k_2}{2} |\xi(s, \pi(s))|_E^2 \sup_{0 \leq t \leq T \wedge \tau} |u(t)|^{q-2}.
\]
Thus we have
\[
\left| \phi(u(s-)) + \xi(s, \pi(s)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s, \pi(s))) \right|_E \\
= \left( 2k_1|\xi(s, \pi(s))|_E \sup_{0 \leq t \leq T \wedge \tau} |u(t)|^{q-1} \right)^{(2-p)+(p-1)} \left( \frac{k_2}{2} |\xi(s, \pi(s))|_E^2 \sup_{0 \leq t \leq T \wedge \tau} |u(t)|^{q-2} \right)^{p-1} \\
\leq K |\xi(s, \pi(s))|_E^p \sup_{0 \leq t \leq T \wedge \tau} |u(t)|^{q-p},
\]
where \(K = (2k_1)^2 - (\frac{k_2}{2})^{p-1} \).

Hence, by Proposition 2.9, we get
\[
\sum_{s \in (0, t \wedge \tau) \cap D(\pi)} \left| \phi(u(s-)) + \xi(s, \pi(s)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s, \pi(s))) \right|_E \\
\leq K \sup_{0 \leq t \leq T \wedge \tau} |u(t)|^{q-p} \sum_{s \in (0, t \wedge \tau) \cap D(\pi)} |\xi(s, \pi(s))|_E^p \\
= K \sup_{0 \leq t \leq T \wedge \tau} |u(t)|^{q-p} \int_0^{T \wedge \tau} \int_Z |\xi(r, z)|_E^p N(dr, dz),
\]

Therefore, we infer
\[
\mathbb{E} \sup_{t \geq 0} |I_2(t)|_E \leq \int_0^{T \wedge \tau} \int_Z \left| \phi(u(r-)) + \xi(r, z)) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, z)) \right|_E N(dr, dz) \\
\leq K \sup_{0 \leq t \leq T \wedge \tau} |u(t)|^{q-p} \int_0^{T \wedge \tau} \int_Z |\xi(s, z)|_E^p N(ds, dz),
\]
where the constant $K$ only depends on $k_1$, $k_2$, $p$ and $q$. Now applying Hölder’s and Young’s inequalities to $I_1(t)$, (4.6), yields

$$
\mathbb{E} \sup_{t \geq 0} |I_1(t)|_E \leq k_1 C \left( \mathbb{E} \left[ \sup_{0 \leq t \leq T^{\land \tau}} |u(t)|_E^{q-1} \right]^{2 \frac{q-1}{q}} \left( \mathbb{E} \left( \int_0^{T^{\land \tau}} \int_Z |\xi(s,z)|_E^p N(ds,dz) \right) \right)^{\frac{q}{2 \frac{q-1}{q}}} \right)^{\frac{1}{q}} 
$$

$$
\leq k_1 C \left( \mathbb{E} \sup_{0 \leq t \leq T^{\land \tau}} |u(t)|_E^q \right)^{\frac{q-1}{q}} \left( \mathbb{E} \left( \int_0^{T^{\land \tau}} \int_Z |\xi(s,z)|_E^p N(ds,dz) \right) \right)^{\frac{q}{2}} \left( \frac{1}{\varepsilon} \right)^{\frac{q-1}{q}} \frac{1}{q} 
$$

$$
= k_1 C \frac{q-1}{q} \varepsilon \mathbb{E} \sup_{0 \leq t \leq T^{\land \tau}} |u(t)|_E^q + k_1 C \left( \frac{1}{\varepsilon^{q-1}} \mathbb{E} \left( \int_0^{T^{\land \tau}} \int_Z |\xi(s,z)|_E^p N(ds,dz) \right) \right)^{\frac{q}{2}}. 
$$

In the same manner for the integral $I_2(t)$ we can see that

$$
\mathbb{E} \sup_{t \geq 0} |I_2(t)|_E \leq K \mathbb{E} \sup_{0 \leq t \leq T^{\land \tau}} |u(t)|_E^{q-p} \int_0^{T^{\land \tau}} \int_Z |\xi(s,z)|_E^p N(ds,dz) 
$$

$$
\leq K \left( \mathbb{E} \sup_{0 \leq t \leq T^{\land \tau}} |u(t)|_E^{q-p} \right)^{\frac{q-p}{q}} \left( \mathbb{E} \left( \int_0^{T^{\land \tau}} \int_Z |\xi(s,z)|_E^p N(ds,dz) \right) \right)^{\frac{q}{2}} \left( \frac{1}{\varepsilon} \right)^{\frac{q-p}{q}} \frac{1}{q} 
$$

$$
\leq K \frac{q-p}{q} \varepsilon \mathbb{E} \sup_{0 \leq t \leq T^{\land \tau}} |u(t)|_E^q + K P \left( \frac{1}{\varepsilon^{q-p}} \mathbb{E} \left( \int_0^{T^{\land \tau}} \int_Z |\xi(s,z)|_E^p N(ds,dz) \right) \right)^{\frac{q}{2}}. 
$$

It then follows that

$$
\mathbb{E} \sup_{0 \leq t \leq T^{\land \tau}} |u(t)|_E^q \leq \left( k_1 C \frac{q-1}{q} + K \frac{q-p}{q} \right) \varepsilon \mathbb{E} \sup_{0 \leq t \leq T^{\land \tau}} |u(t)|_E^q + \left( k_1 C \frac{1}{\varepsilon^{q-1}} + K P \right) \left( \frac{1}{\varepsilon^{q-p}} \right) \mathbb{E} \left( \int_0^{T^{\land \tau}} \int_Z |\xi(s,z)|_E^p N(ds,dz) \right)^{\frac{q}{2}}. 
$$

Now we can choose a suitable number $\varepsilon$ such that

$$
\left( k_1 C \frac{q-1}{q} + K \frac{q-p}{q} \right) \varepsilon = \frac{1}{2}. 
$$

Consequently, there exists $C$ which is independent of $A$ such that

$$
(4.7) \quad \mathbb{E} \sup_{0 \leq t \leq T^{\land \tau}} |u(t)|_E^q \leq C \mathbb{E} \left( \int_0^{T^{\land \tau}} \int_Z |\xi(s,z)|_E^p N(ds,dz) \right)^{\frac{q}{2}}. 
$$

**Case II.** Now suppose $\xi \in \mathcal{M}^p([0,T] \times Z; \mathcal{P}_E)$. Set $R(n, A) = (nI - A)^{-1}, n \in \mathbb{N}$. Then we put $\xi^n(t,\omega,z) = nR(n, A)\xi(t,\omega,z)$ on $[0,T] \times \Omega \times Z$. Since $A$ is the infinitesimal generator of the $C_0$-semigroup $S(t), t \geq 0$ of contractions, by the Hille-Yosida Theorem, $\|R(n, A)\| \leq \frac{1}{n}$ and $\xi^n(t,\omega,z) \in D(A)$, for every $(t,\omega,z) \in [0,T] \times \Omega \times Z$. Moreover, $\xi^n(t,\omega,z) \to \xi(t,\omega,z)$ pointwise on $[0,T] \times \Omega \times Z$. Also, we observe that $|\xi^n - \xi| = |nR(n, A)\xi - \xi| \leq 2|\xi|$. Therefore, it follows by
applying the LDCT that
\[
\mathbb{E} \int_0^T \int_Z 1_{(0,\tau]}(s) |\xi^n(s, z) - \xi(s, z)|_E^p \nu(dz) ds
\]
converges to 0 as \( n \to \infty \), \( \mathbb{P} \)-a.s.. Since the Poisson random measure \( N \) is a \( \mathbb{F} \)-a.s. positive measure and
\[
\mathbb{E} \int_0^T \int_Z 1_{(0,\tau]}(s) |\xi^n(s, z) - \xi(s, z)|_E^p N(ds, dz) = \mathbb{E} \int_0^T \int_Z 1_{(0,\tau]}(s) |\xi^n(s, z) - \xi(s, z)|_E^p \nu(dz) ds,
\]
we see that \( \mathbb{P} \)-a.s.
\[
\int_0^T \int_Z 1_{(0,\tau]}(s) |\xi^n(s, z) - \xi(s, z)|_E^p N(ds, dz) \to 0, \quad \text{as} \ n \to \infty.
\]
Clearly, \( \xi^n \in \mathcal{M}^p([0, T] \times Z; \mathcal{D}(A)) \).

Hence for each \( n \in \mathbb{N} \), we may define a process \( u^n \) by
\[
u^n(t) = \int_0^t S(t-s)\xi^n(s, z) \tilde{N}(ds, dz), \ t \in [0, T].
\]
By Lemma 3.2, the process \( u_n(t) \) can also be formulated in a way of strong solutions so that \( u_n \) is \( E \)-valued càdlàg for each \( n \in \mathbb{N} \). Hence by inequality (4.7), for each \( n \in \mathbb{N} \) and any stopping time \( \tau \geq 0 \) the following inequality holds
\[
\mathbb{E} \sup_{0 \leq t \leq T \land \tau} |u^n(t)|^q \leq C \mathbb{E} \left( \int_0^{T \land \tau} \int_Z |\xi^n(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}}.
\]
On the other hand, since (2.5), we have
\[
\mathbb{E}|u^n(t) - u(t)|_E^p = \mathbb{E} \left| \int_0^t \int_Z \left( S(t-s)\xi^n(s, z) - S(t-s)\xi(s, z) \right) \tilde{N}(ds, dz) \right|_E^p
\leq C_p \mathbb{E} \int_0^T \int_Z |\xi^n(s, z) - \xi(s, z)|_E^p \nu(dz) ds,
\]
we deduce that \( u^n(t) \) converges to \( u(t) \) in \( L^p(\Omega) \) for every \( t \geq 0 \). Moreover, according to (4.7), we know
\[
\mathbb{E} \sup_{t \geq 0} |u^n(t) - u^m(t)|_E^q \leq C \mathbb{E} \left( \int_0^T \int_Z |\xi^n(s, z) - \xi^m(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}}.
\]
Since \( \xi^n(t, \omega, z) \to \xi(t, \omega, z) \) pointwise on \([0, T] \times \Omega \times Z\), we infer that the right hand-side of last inequality converges to 0 as \( n, m \to \infty \). Hence, it is possible to choose a sequence \( \{n_k\}_{k=1}^{\infty} \) of \( \{n\}_{n=1}^{\infty} \) such that
\[
\mathbb{E} \sup_{t \geq 0} |u^{n_{k+1}}(t) - u^{n_k}(t)|_E^q < \frac{1}{k^{2q+2}}.
\]
Hence, on the basis of Chebyshev inequality, we obtain
\[
\mathbb{P} \left\{ \sup_{t \geq 0} |u^{n_{k+1}}(t) - u^{n_k}(t)|_E > \frac{1}{k^2} \right\} \leq k^{2q} \mathbb{E} \sup_{t \geq 0} |u^{n_{k+1}}(t) - u^{n_k}(t)|_E^q < \frac{1}{k^2}.
\]
Thus the series $\sum_{k=1}^{\infty} \mathbb{P}\{\sup_{0 \leq t \leq T} |u_{k+1}(t) - u_k(t)|_E > \frac{1}{k}\}$ is convergent. It follows from the Borel-Cantelli Lemma that with probability 1 there exists an integer $k_0$ such that

$$\sup_{t \geq 0} |u_{k+1}(t) - u_k(t)|_E \leq \frac{1}{k^2}, \text{ for all } k \geq k_0.$$ 

Consequently, the series of càdlàg functions

$$\sum_{k=1}^{\infty} [u_{k+1}(t) - u_k(t)], \quad t \geq 0,$$

converges uniformly on $[0, T]$ with probability 1 to a càdlàg function which we shall define by $\tilde{u} = (\tilde{u}(t))_{t \geq 0}$. In view of Lemma 4.5, it is possible to assume that the process $\tilde{u}$ is separable. Thus, the function $\sup_{t \geq 0} |\tilde{u}(t)|^q$ is also measurable. Moreover, we have

$$E \sup_{t \geq 0} |u_k(t) - \tilde{u}(t)|^q_E \to 0, \text{ as } n_k \to \infty. \quad (4.9)$$

Therefore, by the Minkowski Inequality and (4.7), we have

$$\left[ E \sup_{0 \leq s \leq T \wedge \tau} |\tilde{u}(t)|^q_E \right]^\frac{1}{q} \leq \left[ E \sup_{0 \leq t \leq T \wedge \tau} |\tilde{u}(t) - u_k(t)|^q_E \right]^\frac{1}{q} + \left[ E \sup_{0 \leq t \leq T \wedge \tau} |u_k(t)|^q_E \right]^\frac{1}{q} \leq \left[ E \sup_{0 \leq t \leq T \wedge \tau} |\tilde{u}(t) - u_k(t)|^q_E \right]^\frac{1}{q} + \left[ C E \left( \int_0^{T \wedge \tau} \int_Z |\xi(s, z)|^p_N(d s, d z) \right) \right]^\frac{q}{p}.$$

Note that the constant $C$ on the right hand side of the above inequality does not depend on operator $A$. So the constant $C$ remains the same for every $n$. It follows by letting $n_k \to \infty$ in above inequality that

$$E \sup_{0 \leq t \leq T \wedge \tau} |\tilde{u}(t)|^q \leq C E \left( \int_0^{T \wedge \tau} \int_Z |\xi(s, z)|^p_N(d s, d z) \right)^\frac{q}{p}.$$

Also, by Minkowski inequality we have for every $t \geq 0$,

$$E |\tilde{u}(t) - u(t)|^p_E \leq \left( E |\tilde{u}(t) - u_k(t)|^p_E \right)^\frac{1}{p} + \left( E |u(t) - u_k(t)|^p_E \right)^\frac{1}{p} \leq \left( E |\tilde{u}(t) - u_k(t)|^q_E \right)^\frac{1}{q} + \left( E |u(t) - u_k(t)|^q_E \right)^\frac{1}{q} \leq \left( E \sup_{0 \leq t \leq T} |\tilde{u}(t) - u_k(t)|^q_E \right)^\frac{1}{q} + \left( E |u(t) - u_k(t)|^q_E \right)^\frac{1}{q}.$$

Letting $n \to \infty$, it follows from (4.8) and (4.9) that $u(t) = \tilde{u}(t)$ in $L^q(\Omega)$ for any $t \geq 0$. This shows the inequality (4.2) for $q' = q$. The case $q' > q$ follows from the fact that if Banach space $E$ satisfies Assumption 4.1 for some $q$, then Condition 1 is also satisfied with $q' > q$.

\[\square\]

The following result could be derived immediately from the proof of the above result.

**Corollary 4.1.** Let $E$ be a Banach space satisfying Assumption 4.1. Then the stochastic convolution process $u$ has a càdlàg modification.
Corollary 4.2. Let $E$ be a Banach space satisfying Assumption 4.1. Let $\sqrt{2} \leq p \leq 2$. Then for any $n \in \mathbb{N}$ with $p^n \geq q$, there exists a constant $C = C(E, n)$ such that for every $\xi \in \mathcal{M}^p_{\text{loc}}([0, T] \times Z; \mathcal{P}; E)$ and for every stopping time $\tau > 0$ and $t \geq 0$,

$$
(4.10) \quad \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} |\tilde{u}(s)|^p_{E} \leq C \sum_{k=1}^{n} \mathbb{E} \left( \int_{0}^{t \wedge \tau} \int_{Z} |\xi(s, z)|^p E \nu(\text{d}z) \text{d}s \right)^{\frac{n-k}{k}},
$$

where $\tilde{u}$ is a separable and càdlàg modification of $u$ as before.

The proof of Corollary 4.2 is similar to the proof Lemma 5.2 in Bass and Cranston [2] or of Lemma 4.1 in Protter and Talay [31]. Essential ingredients of that proof are the following two results. The second of them being about integration of real valued processes.

Lemma 4.7. Let $E$ be a martingale type $p$ Banach space, $1 < p \leq 2$, satisfying Assumption 4.1. Let $\tau > 0$ be a stopping time. For any $q' \geq q$, there exists a constant $C$ such that, for all $\xi \in \mathcal{M}^p_{\text{loc}}([0, T] \times Z; \mathcal{P}; E)$, we have

$$
(4.11) \quad \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} \left| \int_{0}^{s} \int_{Z} \xi(r, z) \tilde{N}(\text{d}r, \text{d}z) \right|_{E}^{q'} \leq C \mathbb{E} \left( \int_{0}^{t \wedge \tau} \int_{Z} |\xi(s, z)|^p E N(\text{d}s, \text{d}z) \right)^{\frac{q'}{p}}, \quad t \geq 0.
$$

Proof of Lemma 4.7. This result is a special case of Theorem 4.6 with $S(t) = I, t \geq 0$.

Lemma 4.8. Let $\sqrt{2} \leq p \leq 2$. For any $n \in \mathbb{N}$ there exists a constant $D_n > 0$ such that for any process

$$
(4.12) \quad \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} \left| \int_{0}^{s} \int_{Z} f(r, z) \tilde{N}(\text{d}r, \text{d}z) \right|_{E}^{p^n} \leq D_n \sum_{k=1}^{n} \mathbb{E} \left( \int_{0}^{t \wedge \tau} \int_{Z} |f(s, z)|^p \nu(\text{d}z) \text{d}s \right)^{\frac{n-k}{k}}.
$$

Proof of Lemma 4.8. We shall show this Lemma by induction. The case $n = 1$ follows from [3]. Now we assume that the assertion in the Claim is true for $n - 1$, where $n \in \mathbb{N}$ and $n \geq 2$. We will show that it is still true for $n$. Since by assumption $f \in \mathcal{M}^p([0, T] \times Z; \mathcal{P}; \mathbb{R})$, so both integrals $\int_{0}^{t} \int_{Z} |f(s, z)|^p N(\text{d}s, \text{d}z)$ and $\int_{0}^{t} \int_{Z} |f(s, z)|^p \nu(\text{d}z) \text{d}s$ are well defined as Lebesgue-Stieltjes integrals. Moreover, we have, for every $t \geq 0$ and stopping time $\tau > 0$,

$$
(4.13) \quad \int_{0}^{t \wedge \tau} \int_{Z} |f(s, z)|^p N(\text{d}s, \text{d}z) = \int_{0}^{t \wedge \tau} \int_{Z} |f(s, z)|^p N(\text{d}s, \text{d}z) - \int_{0}^{t \wedge \tau} \int_{Z} |f(s, z)|^p \nu(\text{d}z) \text{d}s \ \mathbb{P}\text{-a.s.}
$$
Since for \( n \geq 2, 2 \leq p^n \leq 2^n \), so the function \(|\cdot|^{p^n}_{\mathbb{R}}\) satisfies Assumption 4.1 with \( q = p^n \). Hence by using first the inequality (4.11) and then (4.13), we infer

\[
\mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} \left| \int_0^s \int_Z f(r, z) \tilde{N}(dr, dz) \right|^{p^n} \leq C \mathbb{E} \left( \int_0^{t \wedge \tau} \int_Z |f(s, z)|^n \tilde{N}(ds, dz) \right)^{p^n-1} 
\]

\[
\leq 2^{p^n-1} C \left\{ \mathbb{E} \left( \int_0^{t \wedge \tau} \int_Z |f(s, z)|^n \tilde{N}(ds, dz) \right)^{p^n-1} + \mathbb{E} \left( \int_0^{t \wedge \tau} \int_Z |f(s, z)|^n \nu(ds, dz) \right)^{p^n-1} \right\} 
\]

Next, by applying the induction assumption to the real valued process

\[
E \left( \sup_{0 \leq s \leq t \wedge \tau} \left| \int_0^s \int_Z f(r, z) \tilde{N}(dr, dz) \right|^{p^n} \right) 
\]

we get

\[
E \sup_{0 \leq s \leq t \wedge \tau} \left| \int_0^s \int_Z f(r, z) \tilde{N}(dr, dz) \right|^{p^n} 
\]

\[
\leq 2^{p^n-1} C \left( D_{n-1} \sum_{i=1}^{n-1} \mathbb{E} \left( \int_0^{t \wedge \tau} \int_Z |f(s, z)|^{p^{i+1}} \nu(ds, dz) \right)^{p^{n-1-i}} 
\quad + \mathbb{E} \left( \int_0^{t \wedge \tau} \int_Z |f(s, z)|^n \nu(ds, dz) \right)^{p^n-1} \right) 
\]

\[
\leq D_n \sum_{k=1}^{n} \mathbb{E} \left( \int_0^{t \wedge \tau} \int_Z |f(s, z)|^{p^k} \nu(ds, dz) \right)^{p^n-k}. 
\]

This proves the validity of the assertion in the Lemma for \( n \) which completes the whole proof. \( \Box \)

Proof of Corollary 4.2. Let us take \( n \in \mathbb{N} \). By applying first Theorem 4.6 and next the equality (4.13) when \( \xi \in \mathcal{M}^p([0, T] \times Z; \tilde{P}; E) \), we deduce that for all \( t \in [0, T] \),

\[
\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|^n_{\mathbb{E}} \leq C \mathbb{E} \left( \int_0^t \int_Z |\xi(s, z)|^n_{\mathbb{E}} \tilde{N}(ds, dz) \right)^{p^n-1} 
\]

\[
\leq 2^{p^n-1} C \mathbb{E} \left( \int_0^t \int_Z |\xi(s, z)|^n_{\mathbb{E}} \tilde{N}(ds, dz) \right)^{p^n-1} 
\quad + 2^{p^n-1} C \mathbb{E} \left( \int_0^t \int_Z |\xi(s, z)|^n_{\mathbb{E}} \nu(ds, dz) \right)^{p^n-1} 
\]

\[
\leq 2^{p^n-1} C D_{n-1} \sum_{k=1}^{n-1} \mathbb{E} \left( \int_0^t \int_Z |\xi(s, z)|^{p^{k+1}} \nu(ds, dz) \right)^{p^{n-1-k}} 
\quad + 2^{p^n-1} C \mathbb{E} \left( \int_0^t \int_Z |\xi(s, z)|^n_{\mathbb{E}} \nu(ds, dz) \right)^{p^n-1} 
\]

\[
\leq C(n) \sum_{k=1}^{n} \mathbb{E} \left( \int_0^t \int_Z |\xi(s, z)|^n_{\mathbb{E}} \nu(ds, dz) \right)^{p^n-k}, 
\]
where we used in the third inequality Lemma 4.7 with \( f \) replaced by real-valued process \( \xi \) such that
\[
|\xi|^p_E \leq \sum_{k=1}^{n-1} \mathcal{M}_{loc}^k([0,T] \times Z; \hat{P}; E).
\]
This completes the proof of Corollary 4.2.

5. Extension to Progressively Measurable Integrands

Corollary 4.2 can be generalized to integrands which are progressively measurable processes. Let us recall that a process \( \xi : [0, T] \times \Omega \times Z \to E \) is \( \mathcal{F} \otimes Z \)-progressively measurable, if \( \xi \) is \( \mathcal{B}F \otimes Z / \mathcal{B}(E) \)-measurable, where, see [33, section 6.5], \( \mathcal{B}F \) is the \( \sigma \)-field consisting of all sets \( A \subset [0, T] \times \Omega \) such that for every \( t \in [0, T] \), the set \( A \cap ([0, t] \times \Omega) \) belongs to the sigma field \( \mathcal{B}_{[0,t]} \otimes \mathcal{F}_t \). Note that \( \mathcal{B}F \otimes Z \) is the \( \sigma \)-field generated by a family of all sets \( A \subset [0, T] \times \Omega \times Z \) such that for every \( t \in [0, T] \), the set \( A \cap ([0, t] \times \Omega \times Z) \) belongs to the sigma field \( \mathcal{B}_{[0,t]} \otimes \mathcal{F}_t \otimes Z \).

For \( p \in [1, \infty) \), the set of all of \( p \)-integrable \( \mathcal{B}F \otimes Z \)-progressively processes \( \xi : [0, T] \times \Omega \times Z \to E \) will be denoted by
\[
\mathcal{M}^p([0,T] \times Z; \mathcal{B}F \otimes Z; E)
\]
and the Banach space of all equivalence classes of \( p \)-integrable \( \mathcal{B}F \otimes Z \)-progressively processes \( \xi : [0, T] \times \Omega \times Z \to E \) will be denoted by
\[
\tilde{\mathcal{M}}^p([0,T] \times Z; \mathcal{B}F \otimes Z; E).
\]

As noted in Remark 2.7, the Itô integral with respect to a compensated Poisson random measure of processes from the class has been introduced in [3], see also [34, Theorem 3.2.27]. The following follows from [34, Theorem 3.2.27].

**Proposition 5.1.** If \( p \in [1, \infty) \) and a progressively measurable process \( \xi : [0, T] \times \Omega \times Z \to E \) belongs to \( \mathcal{M}^p([0,T] \times Z; \mathcal{B}F \otimes Z; E) \) then there exists a sequence of càdlàg step functions \( \xi_n \in \mathcal{M}_{step}^p([0,T] \times Z; \hat{P}; E) \), such that \( \xi_n \to \xi \) in \( \xi \in \mathcal{M}^p([0,T] \times Z; \mathcal{B}F \otimes Z; E) \), as \( n \to \infty \).

**Corollary 5.1.** Let \( E \) be a Banach space satisfying Assumption 4.1. Let \( \sqrt{2} \leq p \leq 2 \) and \( n \in \mathbb{N} \) such that \( p^n \geq q \). Then for every \( \xi \in \cap_{k=1}^n \mathcal{M}_E^p([0,T] \times Z; \mathcal{B}F \otimes Z; E) \) there exists a process \( \tilde{\xi} \) which is a separable and càdlàg modification of the stochastic convolution process \( \tilde{\xi} \) defined, as before, by (3.1), i.e.
\[
\tilde{\xi}(t) = \int_0^t \int_Z S(t-s)\xi(s,z) \tilde{N}(ds,dz), \quad t \in [0,T].
\]
Moreover, there exists a constant \( C = C(E, n) \) independent of \( \xi \), such that for every stopping time \( \tau \) and \( t \in [0,T] \),
\[
\mathbb{E} \sup_{0 \leq s \leq \tau \wedge t} |\tilde{\xi}(s)|_E^p \leq C \sum_{k=1}^n \mathbb{E} \left( \int_0^{\tau \wedge t} \int_Z |\xi(s,z)|_E^p \nu(dz)ds \right)^{p/k},
\]
where \( \tilde{\xi} \) is a separable and càdlàg modification of \( u \) as before.
Proof. By Proposition 5.1, there exists a sequence \( \{ \xi_i : i \in \mathbb{N} \} \subset \cap_{k=1}^\infty \mathcal{M}_{s\text{tep}}^p([0,T] \times Z; \mathcal{F}; E) \) of càdlàg processes convergent to \( \xi \) in \( \cap_{k=1}^\infty \mathcal{M}_{s\text{tep}}^p([0,T] \times Z; BF \otimes Z; E) \). By Theorem 4.6, for every \( i \), the exists a separable càdlàg modification \( \tilde{u}_i \) of the process \( u_i \) being the solution of the Problem

\[(5.3) \quad u_i(t) = \int_0^t \int_Z S(t-s)\xi_i(s,z)\tilde{N}(ds,dz), \quad t \in [0,T],\]

which satisfies

\[(5.4) \quad \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} |\tilde{u}_i(s)|_E^{p^n} \leq C \sum_{k=1}^n \mathbb{E} \left( \int_0^{t \wedge \tau} \int_Z |\xi_i(s,z)|_E^{p^k} \nu(dz)ds \right)^{p^{n-k}}, \quad t \in [0,T], \quad l \in \mathbb{N}.\]

and, for all \( i, j \in \mathbb{N}, \)

\[(5.5) \quad \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} |\tilde{u}_i(s) - \tilde{u}_j(s)|_E^{p^n} \leq C \sum_{k=1}^n \mathbb{E} \left( \int_0^{t \wedge \tau} \int_Z |\xi_i(s,z) - \xi_j(s,z)|_E^{p^k} \nu(dz)ds \right)^{p^{n-k}}, \quad t \in [0,T], \quad l \in \mathbb{N}.\]

Arguing as in the proof of Theorem 4.6 we can conclude the proof. \( \Box \)

6. Final comments

Inequality (1.1) can also be derived by the method used by the second named author and Seidler in [9], see as inequality (4) therein. These authors used the Szekőfalvi-Nagy’s Theorem on unitary dilations in Hilbert spaces. However, this method works only for analytic semigroups of contraction type while the results from the current paper are valid for all \( C_0 \) semigroups of contraction type. Let us now formulate the following result whose proof is a clear combination of the proofs from [9] and [7]. For the explanation of the terms used we refer the reader to the latter work. Similar observation for processes driven by a Wiener process was made independently by Seidler [29].

**Theorem 6.1.** Let \( E \) be a martingale type \( p \) Banach space, where \( 1 < p \leq 2 \). Let \( -A \) be a generator of a bounded analytic semigroup in \( E \) such that for some \( \theta < \frac{1}{2} \pi \), the operator \( A \) has a bounded \( H^\infty(S_\theta) \) calculus. Then, for any \( 0 < q' < \infty \), there exists a constant \( C \) such that for all \( \xi \in \mathcal{M}_{s\text{tep}}^p([0,T] \times Z; \mathcal{F}; E) \) and for every stopping time \( \tau > 0 \), we have

\[\begin{align*}
(\mathbb{B}.1) \sup_{0 \leq s \leq t \wedge \tau} \left| \int_0^s \int_Z S(s-r)\xi(r,z) \tilde{N}(dr,dz) \right|_E^{q'} \leq C \mathbb{E} \left( \int_0^{t \wedge \tau} \int_Z |\xi(s,z)|_E^{p^n} \tilde{N}(ds,dz) \right)^{q'}, \quad t \geq 0.
\end{align*}\]

The following result could be derived immediately from the proof of above theorem.

**Corollary 6.1.** Let \( E \) be a martingale type \( p \) Banach space, where \( 1 < p \leq 2 \). Let \( -A \) be a generator of a bounded analytic semigroup in \( E \) such that for some \( \theta < \frac{1}{2} \pi \) the operator \( A \) has a bounded \( H^\infty(S_\theta) \) calculus. Then, the stochastic convolution process \( u \) defined by (1.1) has càdlàg modification.
7. Appendix

Definition 7.1. A Banach space $E$ with norm $\| \cdot \|$ is of martingale type $p$, for $p \in (1, 2]$ if and only if there exists a constant $C_p(E) > 0$ such that for any $E$-valued discrete martingale $\{M_k\}_{k=1}^n$ the following inequality holds

\[
\mathbb{E}\|M_n\|^p \leq C_p(E) \sum_{k=0}^{n} \mathbb{E}\|M_k - M_{k-1}\|^p,
\]

with $M_{-1} = 0$ as usual.

Remark 7.2. Any separable Hilbert space is of martingale type $2$ with

\[
\mathbb{E}\|M_n\|^2 = \sum_{k=0}^{n} \mathbb{E}\|M_k - M_{k-1}\|^2.
\]

If $E$ and $F$ are isomorphic Banach spaces, then $E$ is of martingale type $p$ if and only if $F$ is of martingale type $p$.

The following definition of 2-smooth Banach spaces in terms of asymptoticity of the modulus of smoothness of the norm can be found in [22] and [23].

Definition 7.3. A Banach space $E$ is $p$-smooth if there exists an equivalent norm defined by the modulus of smoothness of $(E, \| \cdot \|)$

\[
\rho_E(t) = \sup \left\{ \frac{1}{2}(\|x + ty\| + \|x - ty\|) - 1 : \|x\| = \|y\| = 1 \right\}
\]

satisfying $\rho_E(t) \leq Kt^p$ for all $t > 0$ and some $K > 0$.

Remark 7.4. A Banach space is of martingale type $p$ if and only if it is $p$-smooth, see [22]. Hence all spaces $L^q(\mu)$, for $q \in [p, \infty)$ and $q > 1$ with an arbitrary positive measure $\mu$ are of martingale type $p$. Note that any closed subspaces of martingale type $p$ spaces are of martingale type $p$. So the Sobolev spaces $W^{k,q}$, for $q \in [p, \infty)$ and $k > 0$ are of martingale type $p$.

The following Lemma can be found in [32].

Lemma 7.5. A Banach space $E$ is $p$-smooth, $1 < p \leq 2$, if and only if the Fréchet derivative of the norm function $x \mapsto \|x\|^p$ is globally $(p - 1)$-Hölder continuous on $E$.

Lemma 7.6. If a real separable Banach space $E$ satisfies Assumption (4.1), then $E$ is of martingale type $p$, for all $p \in (1, 2]$.

Proof. Let $E$ be a Banach space with norm $\| \cdot \|$. We assume that $q > 2$ and that the function

\[
\psi : E \ni x \mapsto \|x\|^q \in \mathbb{R}
\]

is of $C^2$-class and satisfies the standard assumptions, i.e.

\[
\|\psi'(x)\| \leq C_1\|x\|^{q-1}
\]

and

\[
\|\psi''(x)\| \leq C_2\|x\|^{q-2},
\]
We consider a function 
\[ \phi : E \ni x \mapsto \|x\|^q \in \mathbb{R}. \]
We claim that \( \phi \) is of \( C^1 \) class and of \( C^2 \) class on \( E \setminus \{0\} \), and \( \phi' \) is globally Lipschitz continuous on \( E \).
To see this, observe first by chain rule that for any \( x \in E \setminus \{0\} \),
\[ \phi'(x) = 2 \frac{q}{q} \psi(x) \frac{2}{q} - 1 \psi'(x). \]
Thus,
\[ \|\phi'(x)\| \leq C(\|x\|^q \frac{2}{q} - 1 \|x\|^q - 1) = C\|x\|. \]
In particular, \( \|\phi'(x)\| \to 0 \) as \( x \to 0 \) and thus \( \phi \) is differentiable at 0 and \( d_0 \phi = 0 \).
Applying the chain rule again, we have for \( x \in E \setminus \{0\} \)
\[ \phi''(x) = \frac{2}{q} \left( \frac{2}{q} - 1 \right) \psi(x) \frac{2}{q} - 2 \psi'(x) \otimes \psi'(x) \]
\[ + \frac{2}{q} \psi(x) \frac{2}{q} - 1 \psi''(x) \]
As above, using the assumptions of the derivatives of \( \psi \) we infer that there exists \( C > 0 \) such that
\[ \|\phi''(x)\| \leq C, \ x \in E \setminus \{0\}. \]
For any \( x, y \in E \setminus \{0\} \), by applying Taylor formula we have
\[ \|\phi'(x) - \phi'(y)\| = \|\phi''(\theta)(x - y)\| \leq C\|x - y\|, \]
where the point \( \theta \) lies on the same line segment between \( x \) and \( y \). Hence the first derivative \( \phi' \) is globally Lipschitz continuous. By applying Lemma 7.5, we infer that the Banach space \( E \) is 2-smooth, hence it is of martingale type 2. We know (see [23]) that for \( 1 \leq p < \infty \), there exists a constant \( C \) such that for all \( E \)-valued martingales \( \{M_n\}_{n=0}^\infty \) we have
\[ \mathbb{E} \sup_n \|M_n\|^p \leq C \mathbb{E} \left( \sum_{n=0}^\infty \|M_n - M_{n-1}\|^2 \right)^{p/2}, \]
with \( M_0 = 0 \) as usual. Take \( 1 < p \leq 2 \). It follows that
\[ \sup_n \mathbb{E}\|M_n\|^p \leq \mathbb{E} \sup_n \|M_n\|^p \leq C \mathbb{E} \left( \sum_{n=0}^\infty \|M_n - M_{n-1}\|^2 \right)^{p/2} \leq C \mathbb{E} \left( \sum_{n=0}^\infty \|M_n - M_{n-1}\|^p \right). \]
This shows that \( E \) is of martingale type \( p \). \( \square \)

**Acknowledgements 1.** Preliminary versions of this work were presented at the First CIRM-HCM Joint Meeting on Stochastic Analysis and SPDE’s which was held at Trento (January 2010). The research of the first named author was partially supported by an ORS award at the University of York. Results presented in this article are included in the PhD thesis of the first named author. This work was supported by the FWF-Project P17273-N12. Part of the work was done at the Newton Institute for Mathematical Sciences in Cambridge (UK), whose support is gratefully acknowledged, during the program “Stochastic Partial Differential Equations”. The second named author wishes to thank Clare Hall (Cambridge) for hospitality. The first and second named authors wish to thank
University of Salzburg for hospitality.

Finally, the authors acknowledge that the comments and suggestions of Anna Chojnowska-Michalik made for the PhD thesis of the first named author have also influenced the final presentation of this paper.

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