DIVERGENT FOURIER ANALYSIS USING DEGREES OF OBSERVABILITY

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Mathematical modeling.
We recall that in Relative Set Theory, there is a binary predicate $\langle \cdot \mathsf{st} \cdot \rangle >$, called predicate of relative standardness, which is a preorder on the collection of sets. There are classes of equistandardness: two sets $x$ and $y$ are member of the same class $[x] = [y]$ if $x \mathsf{st} y$ and $y \mathsf{st} x$. Two number $x$ and $y$ are $[^a]$ infinitely close if, for any positive $n \in [a]$, $|x - y| \leq \frac{1}{n}$. Then we write $y^{[^a]} \sim x$. We say that a positive number $X$ is $[^a]$ infinitely large, and we put $y^{[^a]} \sim +\infty$ if $\frac{1}{X} \sim 0$.

We fix once for all $N \in \mathbb{N}$ and two real numbers $\omega(N)$ and $\omega(\omega(N))$ such that

$$\omega(N)^{[N]} \sim +\infty, \quad \omega(\omega(N))^{[\omega]} \sim +\infty$$

From now on, if $x \in \mathbb{R}$ and $y \in \mathbb{R} \cup \{-\infty, +\infty\}$, we simplify

$$x^{[N]} \sim y \text{ in } x \sim y, \quad x^{[\omega(N)]} \sim y \text{ in } x \approx y,$$

$$\omega(N) \text{ in } \omega \text{ and } \omega(\omega(N)) \text{ in } \omega^{[\omega]}.$$
Definition 1.0.1. A set $x$ is said

1. observable if $x \in [N]$,
2. relatively observable if $x \in [w]$
   \[ ||x|| \in \mathbb{R}^N, \]  
   then $x$ is
3. limited if it is relatively observable and there exists an observable real number $\lambda$ such that $||x|| \leq \lambda$.
4. infinite (physically) if $||x||^{[N]} \sim +\infty$,
5. relatively infinite if $||x||^{[\omega]} \sim +\infty$.
6. We say that a relatively observable function $f$ is limited if $f(t)$ is limited for any limited $t$. 


For any limited relatively observable function \( f \), there exists an observable function \( f_0 \) such that, \( f(t) \sim f_0(t) \) for any observable \( t \).

We shall call \( f_0 \) the observable part of \( f \).

For any positive number \( A \), we define the \( N \)-dimensional cubes:

\[
C_A^N = \{(x_1, x_2, \cdots, x_N) \in \mathbb{R}^N : \forall i \in \{1, 2, \cdots N\}, -A \leq x_i \leq A\}
\]

Mostly we shall simplify \( C_A^N \) in \( C_A \).

**Dirac functions**

We fix also two positive and \( C^\infty \) Dirac-functions \( \delta \) and \( \delta[\delta] \) such that \( \delta \) is relatively observable, \( \text{support}(\delta) \) is included in some \( C_{\varepsilon} \), with \( \varepsilon \sim 0 \) and \( \text{support}(\delta[\delta]) \) is included in some \( C_{\varepsilon[\varepsilon]} \), with \( \varepsilon[\varepsilon] \sim 0 \).
Truncated and Regularized functions

If $f$ is a continuous function we define truncated functions $\underline{f}$, $\overline{f}$, and truncated regularized functions $f^\delta$, $f^{\delta[\delta]}$ by:

$$\underline{f} = f \cdot \text{Ind}_{C \omega}, \quad \overline{f} = f \cdot \text{Ind}_{C \omega[\omega]},$$

$$f^\delta = \underline{f} * \delta, \quad f^{\delta[\delta]} = \underline{f} * \delta[\delta].$$

The operation $\star$ is the usual convolution.
Truncated integrals

If $f$ is locally Riemann integrable on $\mathbb{R}^N$ we denote

$$\int f(t) \, dt = \int_{C_\omega} f(t) \, dt, \quad \int f(t) \, dt = \int_{C_\omega[\omega]} f(t) \, dt.$$ 

Truncated and Truncated regularized Fourier-Transforms

We denote

$$\mathcal{F} f = \mathcal{F} f, \quad \mathcal{F}^{-1} f = \mathcal{F}^{-1} f, \quad \mathcal{F}^{-1} f = \mathcal{F}^{-1} f.$$ 

$$\mathcal{F}^\delta f = \mathcal{F} f^\delta, \quad \mathcal{F}^\delta f = \mathcal{F} f^\delta, \quad \mathcal{F}^{-1} f = \mathcal{F}^{-1} f^\delta, \quad \mathcal{F}^{-1} f = \mathcal{F}^{-1} f^\delta.$$ 

We need also the inverse and direct Fourier-transforms in principal value, $\mathcal{F}_P$ and $\mathcal{F}_P^{-1}$ respectively defined, when the limits do exists by:

$$(\mathcal{F}_P F)(x) = \lim_{T \to \infty} \int_{C_T} e^{-2i\pi x \cdot t} F(t) \, dt, \quad (\mathcal{F}_P^{-1} F)(t) = \lim_{X \to \infty} \int_{C_X} e^{2i\pi x \cdot t} F(x) \, dx.$$
Definition 1.0.3. We say that a function $g$ is indiscernible from a function $f$, and we denote $g \equiv f$ if $g(t) \approx f(t)$ for any limited $t$. 
Theorem 1.0.4.

Let $f$ and $g$ be continuous relatively observable functions from $\mathbb{R}^N$ to $\mathbb{R}$, Then

(1) (a) If $F_P^{-1}Ff$ exists, then
$$F^{-1} \circ F f = f = F \circ F^{-1} f.$$ 

(b) With no additional assumption we have
$$F^{-1} \circ F^\delta f \sim f \sim F \circ F^{\delta,-1} f.$$ 

(2) exchange formulae:
$$\int f(t) (Fg)(t) \, dt = \int (Ff)(t) g(t) \, dt,$$
$$\int f(t) (F g)(t) \, dt = \int (F f)(t) g(t) \, dt.$$ 

(3)
$$\int |F f|^2 \approx \int |f|^2.$$
Notation. Let $F$ be a functional transformation of $n$ variables functions, then for any relatively observable $f = (f_1, f_2, \cdots, f_N)$ we denote

$$F^\delta f = F(f) \times \mathcal{F}\delta, \quad F^\delta[\delta] f = F(f) \times \mathcal{F}\delta[\delta].$$
**Definition 1.0.5.** ($\mathcal{F}$-indiscernibility)

(a): We denote $\mathcal{N}$ (resp. $\mathcal{N}'$), the collection of the functions $\varphi \in L^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ such that $\varphi \sim 0$ (resp. $\varphi \approx 0$). If $F$ is a functional transformation of $N$ variables then for any observable (resp. non observable but relatively observable) $f$ we say that $F f$ is $\mathcal{F}$-negligible if there exists $\varphi \in \mathcal{N}$ (resp. $\varphi \in \mathcal{N}'$) such that $F^\delta f = \mathcal{F} \varphi$, (resp.$F^\delta[\delta] f = \mathcal{F} \varphi$). Then we write $F f \not\equiv 0$.

(b): If $F_1$ and $F_2$ are functional transformations, then for any relatively observable $f$ and $g$, we say that $F_1 f$ is $\mathcal{F}$-indiscernible from $F_2 g$ if the function $F(f, g)$ defined by $F(f, g) = F_1 f - F_2 g$ is $\mathcal{F}$-negligible. Then we denote $F_1 f \not\equiv F_2 g$. 
Theorem 1.0.6. Let be $\omega_1$, $\omega_2$, infinitively large positive numbers, and $\omega(\omega)_1$, $\omega(\omega)_2$, relatively infinitively large positive numbers. If we put indexes 1 or 2 to the different Fourier-transform in order to indicate which infinite we use, then for any $f \in \mathbb{R}^N$

1) If $f$ is observable, then $\mathcal{F}_1 f = \mathcal{F}_2 f$, $\mathcal{F}_i f = \mathcal{F}_j f$, $i, j \in \{1, 2\}$.
   If $f$ is not observable but relatively observable, then $\mathcal{F}_1 f = \mathcal{F}_2 f$.

2) If $f \in L^1(\mathbb{R})$ and admits a classical Fourier-transform, $\mathcal{F} f$, then $\mathcal{F} f = \mathcal{F} f$ if $f$ is observable, $\mathcal{F} f = \mathcal{F} f$ if $f$ is not observable but relatively observable.
Unified Denotations.

If $\alpha$ is any relatively observable objects, we denote for any relatively observable function $f$

$$\mathcal{F}_0 f[\text{view from } \alpha] = \begin{cases} \mathcal{F}f, & \text{if } \alpha \text{ is observable;} \\ \underline{\mathcal{F}}f, & \text{if } \alpha \text{ is not observable.} \end{cases}$$

If $\alpha$ is $f$, then we drop the mention $"[\text{view from } \alpha"]"
Theorem 1.0.7. For any continuous relatively observable \( f, g : \mathbb{R}^N \rightarrow \mathbb{C} \), if \( f \ast g \) exists and \( f \ast g \in L^1(\mathbb{R}^N) \), then

\[
\mathcal{F}_0 f \times \mathcal{F}_0 g \overset{\mathcal{F}}{=} \mathcal{F}_0 (f \ast g) \quad [\text{view from } (f, g)]
\]
Theorem 1.0.8. If a relatively observable function $f$ admits a continuous partial derivative $\partial_k f$ then

$$(\mathcal{F}_0 \partial_k f)(x) \xrightarrow{\mathcal{F}} 2i\pi x_k (\mathcal{F}_0 f)(x).$$

Theorem 1.0.9. Let $f$ and $g$ be continuous relatively observable functions. If $\mathcal{F}_0 f \xrightarrow{\mathcal{F}} \mathcal{F}_0 g$ [view from $(f, g)$], then $f = g$. 
Solve the the $\mathcal{F}$-equation,

$$(1 + 2i\pi x)(\mathcal{F}_0 f)(x) \equiv 0,$$

where $f$ is a one variable continuous relatively observable function. We have the obvious solutions $f(t) = K \cdot e^{-t}$, with relatively observable constant $K$, because $f$ is derivable, $f'(t) = -f(t)$ and $(\mathcal{F}_0 f')(x) \equiv 2i\pi x(\mathcal{F}_0 f)(x)$. We prove now that any relatively observable function has this form.

$$(1 + 2i\pi x)(\mathcal{F}_0 f)(x) \equiv 0.$$

$$(\mathcal{F}_0 f)(x) = \frac{1}{1 + 2i\pi x} \times (\mathcal{F}_0 \alpha)(x) = [\mathcal{F}_0(u(t)e^{-t} \ast \alpha(t))](x),$$

with $u = \text{Ind}_{[0,+\infty]}$.

Now, $u(t)e^{-t} \ast \alpha(t) = \int_{-\infty}^{+\infty} u(t-s)e^{-t+s} \alpha(s) \, ds = e^{-t} \int_{-\infty}^{t} e^{s} \alpha(s) \, ds$.

$$|e^{s} \alpha(s)| \leq |e^{t} \alpha(s)| \text{ on } ]-\infty,t].$$

$$A(t) = \int_{-\infty}^{t} e^{s} \alpha(s) \, ds.$$

$$f(t) \sim A(0)e^{-t} \sim f(0)e^{-t}, \quad f(t) \approx A(0)e^{-t} \approx f(0)e^{-t},$$

$f(t)$ and $f(0)e^{-t}$ are observables if $f$ and $t$ are observables, they are relatively observable if $f$ and $t$ are relatively observable. So In the first case we have $f(t) = f(0)e^{-t}$ for any observable $t$, in the second one, $f(t) = f(0)e^{-t}$ for any relatively observable $t$. 15
Solve the differential equation: \( y' + y = \delta \), with relatively observable initial data. By the principle of transfer, the solutions are also relatively observable.

We must have \( \mathcal{F}_0y + 2i\pi x(\mathcal{F}_0y)(x) \equiv (\mathcal{F}_0\delta)(x) \quad [\text{view from } \delta] \). \( \mathcal{F}_0y \) is the sum of relatively observable solutions of

\[
(1 + 2i\pi x)(\mathcal{F}_0y)(x) \equiv 0 \quad [\text{view from } \delta] \quad (1), \quad \text{and} \quad (1 + 2i\pi x)(\mathcal{F}_0y)(x) = (\mathcal{F}_0\delta)(x) \quad (2).
\]

The relatively observable solutions of equation (1) are of the form \( Z(t) = K.e^{-t} \) with relatively observable \( K \). An easy calculation yields to the unique solution of (2),

\[
y_0(t) = e^{-t} \int_{-\infty}^{t} e^{s}\delta(s) \, ds.
\]

So the relatively observables solutions of the differential equation are

\[
y(t) = e^{-t}(\int_{-\infty}^{t} e^{s}\delta(s) \, ds + K)
\]

with relatively observable \( K \).
Let us consider $y_0(t) = e^{-t} \int_{-\infty}^{\xi} e^{s \delta(s)} \, ds$ with observable $t$.

If $t \leq 0$, then $y_0(t) = e^{-t} \int_{-\infty}^{-\xi} e^{s \delta(s)} \, ds = 0$,

if $t > 0$ there exists $\theta \in [-1, 1]$ such that

$$y_0(t) = e^{-t} \int_{-\xi}^{\xi} e^{s \delta(s)} \, ds = e^{-t} e^{\theta \xi} \int_{-\xi}^{\xi} \delta(s) \, ds \sim e^{-t}.$$

$$y_0(0) = e^{\theta' \xi} \int_{-\xi}^{0} \delta(s) \, ds \sim \int_{-\xi}^{0} \delta(s) \, ds,$$ with $\theta' \in [-1, 0]$. $\int_{-\xi}^{0} \delta(s) \, ds \in [0.1]$

c and depends on the shape of the graph of $\delta$. 