Linear Wireless Physical Layer Network Coding based on two-dimensional lattices

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Outline

- Introduction to Wireless Physical-layer Network Coding (WPNC)
- A general network model
- Unambiguous decodability
- Effect of wireless channels
- Constellations based on 2D lattices (Gaussian integers)
- Conclusions
In a multihop network, nodes jointly code incoming data streams instead of switching between them.

- “Butterfly network” example

- Nodes perform bit-by-bit modulo-2 addition on data streams

- Allows both destinations to reconstruct both sources
In wireless network incoming streams cannot so easily be separated

- instead signals add (in complex field), and thus interfere with one another

However we may still be able to decode the network code function from the summed signal

- even without access to separate data symbols

Illustrate with reference to simple example, equivalent to “butterfly”

- the **two-way relay channel**
Two terminals want to exchange data via a relay:

Conventionally this would require 4 time-slots:
It is possible to reduce this to 3 time-slots using network coding:

- In the 3rd time-slot, the relay transmits the modulo-2 sum of data packets A and B.
- Each destination can then reconstruct data by forming the modulo-2 sum with its own data.
We can do better using *Wireless Physical-layer Network Coding* using two phases.

Assume both sources transmit BPSK:

- Map data symbol ‘1’ to signal +1; ‘0’ to -1

- At relay, map signals +2 and -2 to ‘0’; 0 to ‘1’

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>a+b</th>
<th>a⊕b</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-2</td>
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A general network model

Unambiguous decodability

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Conclusions
Model a network with $P$ layers of relays

- with $L(p)$ relays in the $p$th layer
- relay $R_{l}^{(p)}$ connected to relays in the previous layer
- Destination D may be connected to any relay (or source):
  \[
  R_{m}^{(p-1)}, m \in C_{l}^{(p)}
  \]
- Relay $R_{l}^{(p)}$ transmits symbol $S_{l}^{(p)}$, source $S_l$ transmits $s_l$
- Assume w.l.o.g. that D is interested only in $s_1$

\[Q = |C^D|\]
Network coding model

- Relay attempts to decode and forward the function

\[ f_{l}^{(p)} \left( \left\{ s_{m}^{(p-1)} : m \in C_{l}^{(p)} \right\} \right) \]

- where the function can be written:

\[
f_{l}^{(p)} \left( s_{m_{1}}^{(p-1)}, s_{m_{2}}^{(p-1)}, \ldots, s_{m_{|C_{l}|}}^{(p-1)} \right) = \alpha_{m_{1}}^{(p-1)} \odot s_{m_{1}}^{(p-1)} \oplus \alpha_{m_{2}}^{(p-1)} \odot s_{m_{2}}^{(p-1)} \ldots \oplus \alpha_{m_{|C_{l}|}}^{(p-1)} s_{m_{|C_{l}|}}^{(p-1)}
\]

- where \( \alpha_{m_{1}}^{(p-1)}, \alpha_{m_{2}}^{(p-1)}, \ldots, \alpha_{m_{|C_{l}|}}^{(p-1)} \) are the coefficients of the function, and \( m_{1}, m_{2}, \ldots, m_{|C_{l}|} \in C_{l}^{(p)} \) are the members of the relay's connection set in the previous layer.
Algebraic framework

- We have not yet committed ourselves as to the nature of the symbols, coefficients and operators.
- It is clear that the structure must at least be a ring, since two operators are defined, \( \oplus \) and \( \cdot \).
- It has commonly assumed that it further needs to be a field (in which all non-zero elements are invertible), but we will here examine the extent to which that is required.
- Note that we will allow that the input symbols, output symbols and coefficients of the mapping function may each be drawn from a different subset of the ring, respectively:
  - with:
  - This allows for extended cardinality mapping:

\[
S_s, S_f, S_\alpha \subseteq R (q)
\]

\[
S_s \subseteq S_f \subseteq R (q)
\]

\[
q \geq |S_f| > |S_s|
\]
We can relate the vector of outputs of each layer to its inputs via the matrix $A$:

$$s^{(p)} = A^{(p)} s^{(p-1)}$$

We can combine these in cascade, so that:

$$s^{(p)} = A^{(p)} A^{(p-1)} \ldots A^{(1)} s$$

We can write this as a single matrix relating the vector of symbols $s^D$ in the connection set of the destination:

$$s^D = Bs$$

We assume that the destination can (in principle) decode all symbols in its connection set.
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Unambiguous decoding

- For unambiguous decoding of $s_1$ at the destination, $s_D = Bs$
must be different for different $s_1$, regardless of $s_2$ to $s_{L(0)}$.

\[
B \begin{bmatrix}
S_1 \\
S_2 \\
\vdots \\
S_{L(0)}
\end{bmatrix} \neq B \begin{bmatrix}
S'_1 \\
S'_2 \\
\vdots \\
S'_{L(0)}
\end{bmatrix}, \forall s_1 \neq s_1', s_2', s_2', \ldots, s_{L(0)}', s'_{L(0)}
\]

- Suppose there did exist some $s_1, s_1'$

where $B(l)$ denotes the $l$th column of $B$, and '-' denotes addition of additive inverses.

- For the general ring, this could be satisfied if either

1) LHS is zero, or

2) LHS is non-zero, and RHS is equal to it
Case (1)

- Since \(s_1\) and \(s_1'\) cannot both be zero, this can be satisfied only if:

  a) \(B_{(1)} = 0\)

  b) All non-zero elements \(b\) of \(B_{(1)}\) are either:

     i. zero divisors, such that

     $$\exists s \neq 0 \in S_s \text{ such that } bs = 0$$

     ii. or

     $$\exists s \neq s' \in S_s \text{ such that } bs = bs'$$

- If the elements of \(B\) are from an integral domain, (i) is not satisfied
Case (2) could occur only if \( B_1 \) is not linearly independent of all other columns.

Hence we can be sure of unambiguous decodability of \( s_1 \) provided:

**A.** All elements of \( B_1 \) are uniquely invertible, and are not zero divisors

**B.** \( B_1 \) is linearly independent of all other columns.

These are sufficient conditions: they may not always be necessary.

Note that **A.** is always satisfied if the symbols and coefficients form a field.
Simplest example:

- Two-way relay channel is equivalent to butterfly network

For $D_1$, we can write:

$$s_1^{(1)} = A_1^{(1)} s^{(0)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \end{bmatrix} \begin{bmatrix} s_1^{(0)} \\ s_2^{(0)} \end{bmatrix}$$

where $a_{11}$ and $a_{12}$ are the coefficients of the network code function at $R_1^{(0)}$

Symbol vector in $D_1$ connection set is:

$$s^D = \begin{bmatrix} s_1^{(1)} \\ s_2^{(0)} \end{bmatrix} = Bs = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_1^{(0)} \\ s_2^{(0)} \end{bmatrix}$$

$2^{nd}$ column of $B$ is independent of $1^{st}$

$a_{11}$ must be uniquely invertible/non zero-divisor, according to our sufficient condition
Recall however that:

\[ S_s, S_f, S_\alpha \subseteq \mathbb{R}(q) \]

i.e. that extended cardinality mapping is permitted:

For butterfly network:

\[
\begin{bmatrix}
    a_{11} & a_{12} \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    s_1 \\
    s_2
\end{bmatrix}
\neq
\begin{bmatrix}
    a_{11} & a_{12} \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    s'_1 \\
    s'_2
\end{bmatrix},
\forall s_1 \neq s'_1, s_2, s'_2 \in S_s
\]

For:

\[
\begin{bmatrix}
    a_{11} & a_{12} \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    s_1 \\
    s_2
\end{bmatrix}
= \begin{bmatrix}
    a_{11} & a_{12} \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    s'_1 \\
    s'_2
\end{bmatrix}
\]

\[a_{11}s_1 + a_{12}s_2 = a_{11}s'_1 + a_{12}s'_2\]

\[a_{11}s_1 = a_{11}s'_1, s_1 \neq s'_1 \in S_s\]

For this to be false it is sufficient that \(a_{11}\) is uniquely invertible, but not necessary

since

\[ \begin{bmatrix}
    s_1, s'_1
\end{bmatrix} \]

may be restricted to a subset of

\[ \mathbb{R}(q) \]

\[q \geq |S_f| > |S_s|\]
For unambiguous decodability, multiplication by $a_{11}$ must be a bijective or injective mapping from $S_S$ to $S_f$.

The set of products $a_{11}s_1, s_1 \in S_S$ is (at least a subset of) an ideal $I(a_{11})$ of $S_f$ generated by $a_{11}$.

Hence a necessary condition is that:

- A hypothesis is that it is possible to choose $S_S$ as a suitable subset such that this is also sufficient.

$$|I(a_{11})| \geq |S_S|$$
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A wireless signal used to transmit a data symbol using **linear modulation** can be represented as a complex number representing amplitude and phase of the transmitted carrier signal.

The set of signals $x$ corresponding to the set of symbols $S$ is called the **constellation** $M : s \rightarrow x$.

Wireless channels subject to **multipath fading** may be represented by multiplication by a complex Gaussian random number $h$:

$r$ is a complex Gaussian random number representing noise at the receiver.

$$y = hx + n$$
Consider the effect of fading of the source-relay channels.

Consider a pair of nodes which transmit a pair of modulated data symbols.

We say that a **singular fade state** (SFS) occurs if the channel coefficients are such that:

However, we say that the singular fade state is **resolved** if for all symbol pairs \( (s_A, s_B) \) and \( (s'_A, s'_B) \) such that \( x_{\{A,B\}} = M \left( s_{\{A,B\}} \right) \) and \( h_A \) and \( h_B \) are such that:

\[
y\left( s_A, s_B \right) = h_A x_A + h_B x_B = y\left( s'_A, s'_B \right) = h_A x'_A + h_B x'_B, \quad s_A \neq s'_A, s_B \neq s'_B
\]

The network coding function gives the same symbol:

\[
y\left( s_A, s_B \right) = y\left( s'_A, s'_B \right)
\]
Singular fade states occur if

\[ \frac{h_B}{h_A} = h_{re} = \frac{x'_A - x_A}{x_B - x'_B} \]

where \( x' \) are signals from the constellation.

For QPSK:

\[ h_{re} = \begin{cases} 
\pm 1 \\
\pm j \\
\pm 1 \pm j \\
2 \\
0 
\end{cases} \]
Resolving singular fade states

- Relay chooses linear mapping so that symbol pairs that give same signal map to same network coded symbol
- Assumes relay (not necessarily sources) knows relative fadestate

$h_{re} = 1$

Ring $q = 4$

coefficients $\alpha_1 = \alpha_2 = 1$
Nearly singular fading

- In the presence of noise, fading does not need to be precisely singular to prevent reliable decoding of network code function.
- If two symbol pairs which result in different function results are close together, this will increase equivocation.
- Hence we are potentially interested in **nearly singular fading**.
- i.e.

\[
y(s_A, s_B) \approx y(s_A', s_B')
\]

\[
|y(s_A, s_B) - y(s_A', s_B')| < \varepsilon
\]
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- **Constellations based on 2D lattices (Gaussian integers)**
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Let the source symbol set be a square size $(2L+1)^2$ set of Gaussian integers.

Can also define this symbol set as a nested lattice.

Transmitted signal constellation is a scaled (and possibly translated) version of the Gaussian integers.

Hence effect of channel can be directly modelled by multiplying the symbol by the channel coefficient.

$$S = \mathbb{Z}[i] \cap \left\{ z : \left| \text{Re}[z] \right| \leq L \& \left| \text{Im}[z] \right| \leq L \right\}$$
Fine lattice is defined as:

\[ \Lambda_f = \{ s : ka + lb \} , k, l \in \mathbb{Z} , a = 1 , b = i \]

Coarse lattice is a sub-lattice of this:

\[ \Lambda_c = \{ s : ka_c + lb_c \} , k, l \in \mathbb{Z} , a_c = (2L+1) , b_c = (2L+1)i \]
Singular fading

- SFS occur when:

\[ h_A x_A + h_B x_B = h_A x'_A + h_B x'_B \]
\[ h_A s_A + h_B s_B = h_A s'_A + h_B s'_B \]
\[ h_A (k_A + l_A i) + h_B (k_B + l_B i) = h_A (k'_A + l'_A i) + h_B (k'_A + l'_A i) \]

which forms a Diophantine equation

- We are interested in both the values of \( h_A \) and \( h_B \) which yield solutions, and in the solutions for \( k, l \) etc

\[ h_{re} = \frac{h_B}{h_A} = \frac{k'_A - k_A + (l'_A - l_A) i}{k_B - k'_B + (l'_B - l'_B) i} = \frac{k'' + l'' i}{k'' + l'' i} \]

- In particular we are interested in the probability of singular or nearly singular fading occurring

- Relative fade coefficients \( h_{re} \) for SFS form set of complex rational

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Singular fade coefficients
For the Gaussian integers, network code functions for singular fade states may be simply the corresponding channel coefficients.

Hence all singular fade states (within some scaling):

\[ f(s_A, s_B) = h_A s_A + h_B s_B \]

and thus whenever two symbol pairs give the same received signal they also give the same function:

\[ y(s_A, s_B) = h_A x_A + h_B x_B = h_A s_A + h_B s_B = f(s_A, s_B) \]
Reducing cardinality

- This greatly increases cardinality of symbol set, and hence rate relay must transmit
- e.g. for $h_A = h_B = 1$
Hence we reduce resulting signal set modulo original coarse lattice.

In butterfly network we can then always recover required symbol since Gaussian integers with addition and multiplication modulo lattice forms a ring and coefficient 1 is invertible within the ring.

However there are SFS which do not correspond to coefficients in the ring.
Consider $L = 4$

Consider the SFS $h_A = 1, h_B = 2$

Now 2 is not invertible on the ring

Hypothesis – these cases can be resolved allowing extended cardinality

that is, by using a different coarse lattice
Outlined general model of multilayer relay network using wireless physical-layer network coding

Gave conditions on network code functions at relays for unambiguous decodability of sources at destination
  - coefficients should be invertible on the ring

Defined singular fading, and gave conditions for resolution of singular fading

Considered constellations and network code functions based on Gaussian integers modulo a lattice
  - provides a natural relationship between network code functions and channel fading
  - in principle SFS are automatically resolved

SFS are solutions of Diophantine equation

However in some cases coefficients are not invertible