Young measures as probability distributions of Loeb spaces

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If \( u^\epsilon(x) \in L^\infty(\Omega) \), then we know there is a subsequence (we still label as \( u^\epsilon(x) \)) and some function \( u(x) \) such that \( u^\epsilon(x) \) weakly star converges to \( u(x) \), i.e.

\[
u^\epsilon(x) \rightharpoonup^* u(x).
\]

For any continuous function \( \phi \), we have subsequence

\[
\phi(u^\epsilon(x)) \rightharpoonup^* l.
\]

What’s the relation between \( \phi(u(x)) \) and \( l \)?
Answer: In general, $\phi(u(x)) \neq I$.
Because it is weak star convergence here.

Is there any relation between $\phi(u(x))$ and $I$?
Young measures can say something here. Theorem about Young measure is

**Theorem**

Let $\Omega \subset \mathbb{R}^n$, $K \subset \mathbb{R}^m$ be bounded open sets and $u^\varepsilon : \Omega \to \mathbb{R}^m$ be measurable functions satisfying $u^\varepsilon \in K$ a.e.. Then there exists a family of probability measures $\nu_x \in \text{Prob}(\mathbb{R}^m)$, $x \in \Omega$ such that

$$\text{supp} \nu_x \subset \bar{K}, \quad x \in \Omega.$$ 

and for any continuous function $\phi : \mathbb{R}^m \to \mathbb{R}$, there is a subsequence (still labeled) $u^\varepsilon$ satisfying

$$w^* - \lim \phi(u^\varepsilon(x)) = \int \phi(y) d\nu_x(y).$$

(1)
From above theorem, we can know

\[ u(x) = w^* - \lim (u^\varepsilon(x)) = \int y dv_x(y) \]

so

\[ \phi(u(x)) = \phi(\int y dv_x(y)) \]

and

\[ l = w^* - \lim \phi(u^\varepsilon(x)) = \int \phi(y) dv_x(y) \]

From here, we can see \( \phi(u(x)) \) and \( l \) can be connected by the Probability measures \( v_x \). It is also quite obvious that in general,

\[ \phi(u) = \phi(\int y dv_x(y)) \neq \int \phi(y) dv_x(y) = l \]
Once we know the Young measure, we can get the weak star limit of $\phi(u^\varepsilon(x))$ for any continuous function $\phi(y)$.

If the Young measures $\nu_x$ are $\delta_{u(x)}$, a.e. $x \in \Omega$, then the weak star convergence becomes the strong convergence. i.e. convergence a.e. Then in many cases we can use the Dominate convergence theorem and pass the limit.
In PDE, Young measures play very important role in the Compensated Compactness method.

It is a very useful tool to prove the global existence of $L^\infty$ weak entropy solutions for some PDE equations system.

Young measure is also very important for the relaxed control.
It is very hard to write down the Young measure explicitly. It is very abstract in the following sense:

Young measures satisfy:

$$\lim_{\varepsilon} \int f(x)\phi(u^{\varepsilon}(x))dx = \int \left( f(x) \int \phi(y)d\nu_{x}(y) \right)dx,$$

(2)

for any $f(x) \in L^1(\Omega)$, any $\phi(y)$ continuous function on compact support.

If we want to describe the Young measure, we have to depend on $f(x)$ and $\phi(y)$. It doesn’t give us the explicit form of the Young measure, it is in the integral.
So it seems **hopeless** to write down the Young measure explicitly using our **standard analysis**.

While in the **nonstandard space**, this job can be done, i.e. we can represent the Young measure in an very explicit form.
Previous result: Cutland, represent Young measures by internal hyperfinite function. 1983.

Here we use a different approach.

Main theorem:
Let $U : * \mathbb{R}^m \rightarrow * Y$ be an internal Lebesgue measurable function with $Y$ a compact subset of $\mathbb{R}^n$, then there is a Young measure $\nu$ such that

$$\int_X f(x) dx \int_Y \phi(y) d\nu_x(y) = \text{st}(\int_{* \mathbb{R}^m} \phi(U(x)) * f(x) dx)$$

for any $f \in L^1(dx)$ and $\phi \in C_Y$.

For almost all points $x_0$ in $\mathbb{R}^m$ and any bounded measurable set $B \subset \mathbb{R}^m$ with positive measure, there exists a positive infinitesimal $\rho_0$ such that if $\rho \geq \rho_0$ is an infinitesimal, then the Young measure $\nu_{x_0}$ is the probability distribution of $^0U(x)$ defined on $x_0 + \rho * B = \{x \in * \mathbb{R}^m/ x = x_0 + \rho b, b \in * B\}$ with the natural Loeb probability measure and

$$\int_Y \phi(y) d\nu_{x_0}(y) \approx \frac{1}{\rho^m |B|} \int_{\rho * B} * \phi(U(x_0 + b)) db$$

for any $\phi \in C_Y$, where $|B|$ is the Lebesgue measure of $B$. 
A direct and simple application to the weakly convergent function sequences $\{u^n(x)\}_{n=1}^{\infty}$ is following:
Corollary 1

Let $Y$ be a compact subset of $\mathbb{R}^n$, let $\{u^n\}_{n=1}^{\infty} \colon \mathbb{R}^m \to Y$ be measurable functions and there is a Young measure $\nu$ corresponding to $\delta_{u^n(x)}$, then for any $\omega \in^{*} \mathbb{N} \setminus ^{*}(\mathbb{N})$, we have

$$\int_{\mathbb{R}^m} f(x)dx \int_{Y} \phi(y)d\nu_{x}(y) = \text{st}(\int_{\mathbb{R}^m}^* \phi(u^\omega(x))^* f(x)dx)$$

for any $f \in L^1(dx)$ and $\phi \in C_Y$.

For almost all points $x_0$ in $\mathbb{R}^m$ and any bounded measurable set $B \subset \mathbb{R}^m$ with positive measure, there exists a positive infinitesimal $\rho_0$ such that if $\rho \geq \rho_0$ is an infinitesimal, then the Young measure $\nu_{x_0}$ is the probability distribution of $u^\omega(x)$ defined on $x_0 + \rho^* B = \{x \in ^* \mathbb{R}^m / x = x_0 + \rho b, b \in ^* B\}$ with the natural Loeb probability measure and for any $\phi \in C_Y$,

$$w^{*}\lim_{n \to \infty} \phi(u^n)(x_0) = \int_{Y} \phi(y)d\nu_{x_0}(y) \approx \frac{1}{\rho^m |B|} \int_{\rho^* B}^* \phi(u^\omega(x_0+b))db$$

where $|B|$ is the Lebesgue measure of $B$. 
By Corollary 1, we can compute the Young measure of some weakly convergent sequence.

Example 1:

\( x \in \mathbb{R} \),

\( I(x) \) is the periodic function of folding line.

\( u^n(x) = I(nx) \)

![Graph of \( u^n(x) \)](image-url)

**Figure:** \( u^n(x) \)
What’s the Young measures corresponding to this $u^n(x)$?

$\nu_{x_0}$ is the probability distribution of $0^u(x)$ defined on $x_0 + \rho^*B = \{ x \in \mathbb{R}^m / x = x_0 + \rho b, b \in *B \}$ with the natural Loeb probability measure when $\rho$ is an infinitesimal big enough.

Therefore for any $x_0$, the $\nu_{x_0}$ is the uniform distribution on (-1, 1).
When $\phi(y) = y^2$, 

It is easy to compute 

\[
(u^n(x))^2 \to \int y^2 d\nu_x = \frac{1}{2} \int_{-1}^{1} y^2 dy = \frac{1}{3}.
\]
Example 2
\[ u^n(x) = \sin(nx) \]

Figure: \( \sin(nx) \)
What’s the Young measures corresponding to this $u^n(x)$?

$u^\omega(x) = \sin(\omega x)$

![Graph of $\sin(\omega x)$](image)

**Figure:** $\sin(\omega x)$

Therefore the density of the Young measures $\nu_x$ is

$$\frac{1}{\pi} (\arcsin y)' = \frac{1}{\pi} \frac{1}{\sqrt{1 - y^2}}$$
We hope to use $\epsilon - \delta$ to describe the above theorem.
By transfer Principle, we have the following:

**Corollary**

\( Y \) is a compact subset of \( \mathbb{R}^n \), Let \( \{u^n\}_{n=1}^\infty : \Omega \to Y \) be measurable functions and there is a Young measure \( \nu \) corresponding to \( \delta_{u^n(x)} \).

For almost all points \( x_0 \) in \( \Omega \) and any bounded measurable set \( B \subset \mathbb{R}^m \) with positive measure, \( \forall \epsilon > 0, \forall \delta > 0, \forall \phi \in C_Y, \exists n \in \mathbb{N}, \forall m \geq n, \exists \delta_0, \delta_1 \) such that \( \delta_0 < \delta_1 \leq \delta \), for any \( \tilde{\delta} \in [\delta_0, \delta_1] \), we have

\[
\left| \int_Y \phi(y) d\nu_{x_0}(y) - \frac{1}{|\tilde{\delta}B|} \int_{\tilde{\delta}B} \phi(u^m(x_0 + b)) db \right| < \epsilon
\]
But here the estimate depends on $\phi$. For different $\phi$, we may have different set $x_0 + \tilde{\delta}B$. So we can not get information for Young measure $\nu_{x_0}$. Furthermore, it suggests that we even couldn’t get that the Young measure is the weak limit of some sequence of probability measure in the standard analysis.

But in Theorem 1, we can find a set $x_0 + \rho^* B$ which is independent of $\phi$, so that we can extract Young measure $\nu_{x_0}$ from there.
It is well known in the theory of ordinary differential equations that Lipschitz condition implies uniqueness of the solutions and continuous dependence of the solutions to the initial data and parameters.

It is proved by using nonstandard analysis in [L1] that the uniqueness implies continuous dependence of the solutions to the initial data and parameters. More precisely we have (Theorem 5.6.5 in [L1]):

**Theorem**

If for any \( a \in \mathbb{R}^m, \mu \in \mathbb{R}^n, T \in [0, \infty) \), there is a unique \( x(a, t, \mu) \in \mathbb{R}^m \) for \( t \in [0, T] \) such that

\[
x(a, 0, \mu) = a, \quad \frac{dx}{dt}(a, t, \mu) = X(x(a, t, \mu), \mu)
\]

where \( X(x, \mu) \in \mathbb{R}^m \) is a continuous function on \( \mathbb{R}^m \times \mathbb{R}^n \), then \( x(a, t, \mu) \) is a continuous function on \( \mathbb{R}^m \times [0, \infty) \times \mathbb{R}^n \).
Here we prove a similar result for relaxed controls. i.e. the parameters are Young measures $\nu$.

**Theorem**

Let $Y$ be any compact matrix space, $dt$ the Lebesgue measure on $[0, 1]$, $f(t, x, y)$ a Caratheodory function on $[0, 1] \times (\mathbb{R}^m \times Y)$ such that $|f(t, x, y)| \leq \varphi(t)(1 + |x|)$ for some integrable function $\varphi$. Suppose that for any $a \in \mathbb{R}^m$ and $\nu \in M(dt, Y)$, there is a unique absolutely continuous function $x(a, \nu, t) : [0, 1] \rightarrow \mathbb{R}^m$ such that

$$x(a, \nu, t) = a + \int_0^t d\tau \int_Y f(\tau, x(a, \nu, \tau), y) d\nu_\tau(y).$$

then the map $\mathbb{R}^m \times M(dt, Y) \rightarrow C([0, 1], \mathbb{R}^m)$ given by $(a, \nu) \rightarrow x(a, \nu, \cdot)$ is a continuous function, where $C([0, 1], \mathbb{R}^m)$ is the Banach space of continuous functions $\phi : [0, 1] \rightarrow \mathbb{R}^m$ with norm $\sup_t |\phi(t)|$. 

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Proof of Theorem 1

The set $M(dx, Y)$ of Young measures can be identified with the bounded linear operators $C_Y \longrightarrow L^\infty(dx)$ between the two Banach spaces given by $\phi \longrightarrow \nu_\phi$ such that $\phi \geq 0$ implies $\nu_\phi \geq 0$ and $\phi \equiv 1$ implies $\nu_\phi \equiv 1$. If $U : X \longrightarrow Y$ is a measurable function, the corresponding $\nu_\phi$ is given by $\nu_\phi(x) = \phi(U(x))$. Now let $U$ be an internal measurable function $*X \longrightarrow *Y$, then $*\phi(U(x))$ is an element in $*L^\infty(dx)$ with finite norm. The compactness of the unit ball of $L^\infty(dx)$ with weak star topology (Banach-Alaoglu theorem), and the nonstandard criterion of compactness of Robinson (see [Cu1] or [SL]) imply that there is a unique $\nu_\phi \in L^\infty(dx)$ such that

$$\int_X \nu_\phi(x)f(x)dx \simeq \int_{*X} *\phi(U(x)) *f(x)dx$$

for any $f \in L^1(dx)$. It is easy to see that there is a Young measure $\nu$ with
\[ \nu_\phi(x) = \int_Y \phi(y) d\nu_x(y), \quad a.e. \]

and the density of \( F(dx, Y) \) in \( M(dx, Y) \) implies that every \( \nu \in M(dx, Y) \) can be obtained in this way from some internal measurable function \( U \).

Take a countable dense subset \( \{\phi_k\} \) of \( C_Y \), then there is a subset \( E \) of \( X = \mathbb{R}^m \) such that \( X - E \) has measure zero and any \( x_0 \in E \) is a Lebesgue point for any \( \nu_{\phi_k} \). We claim that for any \( \phi \in C_Y \), \( x_0 \) is also a Lebesgue point of \( \nu_\phi \), i.e.

\[ \lim_{h \to 0} \int_{Q_h} |\nu_\phi(x + x_0) - \nu_\phi(x_0)| dx = 0 \]

where \( Q_h = \{x \in \mathbb{R}^m/ |x_i| < h/2, i = 1, 2, \cdots, m\} \). For any \( \varepsilon > 0 \), take a \( \phi_k \) such that \( |\phi(y) - \phi_k(y)| < \varepsilon \) for any \( y \in Y \). Then \( |\nu_\phi(x) - \nu_{\phi_k}(x)| < \varepsilon \) for any \( x \in X \).
Hence
\[
\lim_{h \to 0} \frac{1}{h^m} \int_{Q_h} |\nu_\phi(x + x_0) - \nu_\phi(x_0)|dx < 2\varepsilon
\]

The claim is proved.

According to Theorem 1 in [L2], for any \(x_0 \in E\) and bounded measurable set \(B\) in \(X\) with positive measure, there is a positive infinitesimal \(\rho_{0,k}\) such that for any infinitesimal \(\rho \geq \rho_{0,k}\),

\[
\nu_{\phi_k}(x_0) \simeq \frac{1}{\rho^m |B|} \int_{\rho^* B} \phi_k(U(x_0 + b))db
\]

By Robinson lemma, there is an infinitesimal \(\rho_0 \geq \rho_{0,k}\) for any \(k\).

Then it is easy to see that for any \(\phi \in C_Y\) and infinitesimal \(\rho \geq \rho_0\),

\[
\nu_\phi(x_0) \simeq \frac{1}{\rho^m |B|} \int_{\rho^* B} \phi(U(x_0 + b))db
\]
Therefore

\[ \nu_{\phi}(x_0) = \int_{\rho^* B} \phi(\circ U(x_0 + b)) d_L b = \int_Y \phi(y) d\nu_{x_0}(y) \]

This shows that \( \nu_{x_0} \) is the probability distribution of the function \( \circ U(x) \) defined on the Loeb probability space \( x_0 + \rho^* B \), and the proof is complete.
Thanks!