On Some Paradoxes of Measure Theory  
from the Class Standpoint

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Abstract.  
We contend that a treatment of paradoxes in terms of classes is more elucidative than the customary as well as more illuminating of the dichotomy between Continuity and Choice, eg:
Consider the set of real numbers $\mathbb{R}$ as part of a wider proper class of surreal numbers [1]. All nonstandard models of $\mathbb{R}$ can also be embedded in this class. As for the discrete distribution of real numbers on the surreal line, all sets in ZFC are discrete. This helps to explain easily some paradoxical results of Lebesgue measure theory in $\mathbb{R}^n$.

In a usual description of nonmeasurable sets, the unit interval $[0, 1]$ is divided into a countable union $\bigcup_{n \in \mathbb{Z}} A_n$ of congruent $A_n$’s using the axiom of choice [2]. The sets $A_n$ cannot have a definite Lebesgue measure if we demand that all $A_n$ have equal measures; but this result is based on the confusion of the set of real numbers of $[0, 1]$ and the whole proper class $[0, 1]$ which has no cardinal number.

We can define a ”measure” $\mu$ on $[0, 1]$ in a more reasonable way as follows. For all sets $A \mu(A) = 0$; for all intervals $(a, b)$ regarded as proper classes $\mu(a, b) = b - a$. This definition of $\mu$ just gives $\sum A_n = 0$. Problem: What can be said about $\mu(A_n)$? What kind of additivity can one expect from $\mu$?

There is also no surprise with the Cantor set[2] simultaneously having the cardinality of continuum and measure zero. The measure $\mu$ extends this property of nowhere dense sets to all sets.

The famous Banach-Tarski’s procedure [3] is often an occasion to call in question the axiom of choice: the unit ball $B$ in $\mathbb{R}^3$ is divided into six sets, of which some are nonmeasurable, that can be reassembled into two unit balls by means of shifts and rotations. This construction is not stranger than the following well known one. Divide the set $\mathbb{N}$ of natural numbers into two sets of odd and even numbers. Shifting odd numbers by one to the right and contracting two resulting sets of even numbers, we get two sets $\mathbb{N}$. Roughly speaking, from one set of measure zero we get two such sets. We see no mystery about it. And we see no mystery about the Banach-Tarski’s procedure from the class standpoint if we discern the set $B$ and the whole class containing $B$.

It may be interesting also to look at the problem of measurable cardinals in terms of classes.